Engineering Electromagnetics

W.H. Hayt Jr. and J. A. Buck

Chapter 7:

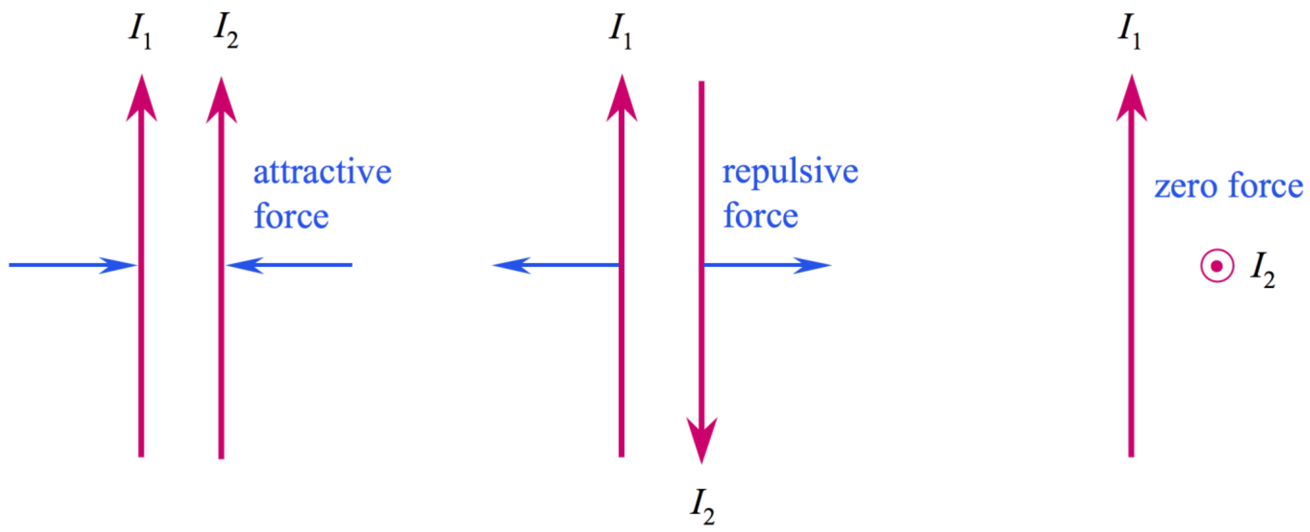
The Steady Magnetic Field

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ϵ and σ	μ
Governing equations		
• Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\epsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

Motivating the Magnetic Field Concept: Forces Between Currents

Magnetic forces arise whenever we have charges in motion. Forces between current-carrying wires present familiar examples that we can use to determine what a magnetic force field should look like:

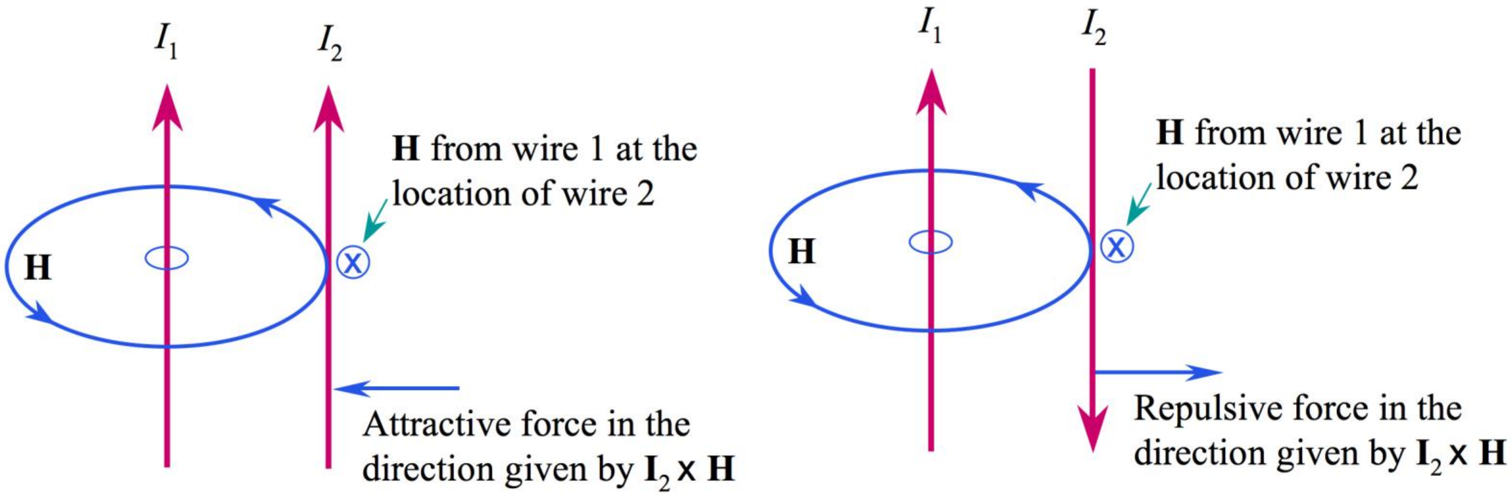
Here are the easily-observed facts:



How can we describe a force field around wire 1 that can be used to determine the force on wire 2?

Magnetic Field

The geometry of the magnetic field is set up to correctly model forces between currents that allow for any relative orientation. The magnetic field intensity, **H**, circulates around its source, I_1, in a direction most easily determined by the right-hand rule: Right thumb in the direction of the current, fingers curl in the direction of **H**



Note that in the third case (perpendicular currents), I_2 is in the same direction as **H**, so that their cross product (and the resulting force) is zero. The actual force computation involves a different field quantity, **B**, which is related to **H** through $\mathbf{B} = \mu_0\mathbf{H}$ in free space. This will be taken up in a later lecture. Our immediate concern is how to find **H** from any given current distribution.

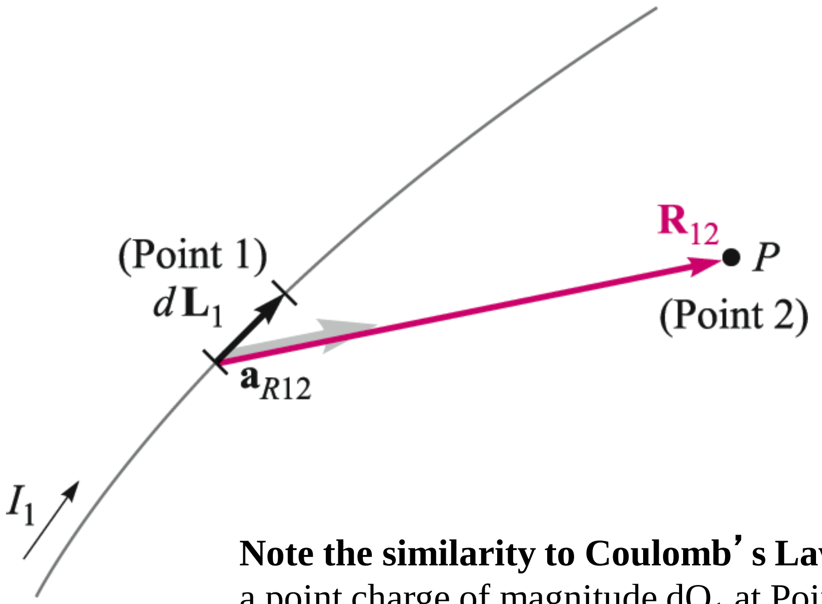
Biot-Savart Law

The Biot-Savart Law specifies the magnetic field intensity, **H**, arising from a "point source" current element of differential length d**L.**

$$\boxed{d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}}$$

The units of **H** are [A/m]

Note in particular the inverse-square distance dependence, and the fact that the cross product will yield a field vector that points into the page. This is a formal statement of the right-hand rule

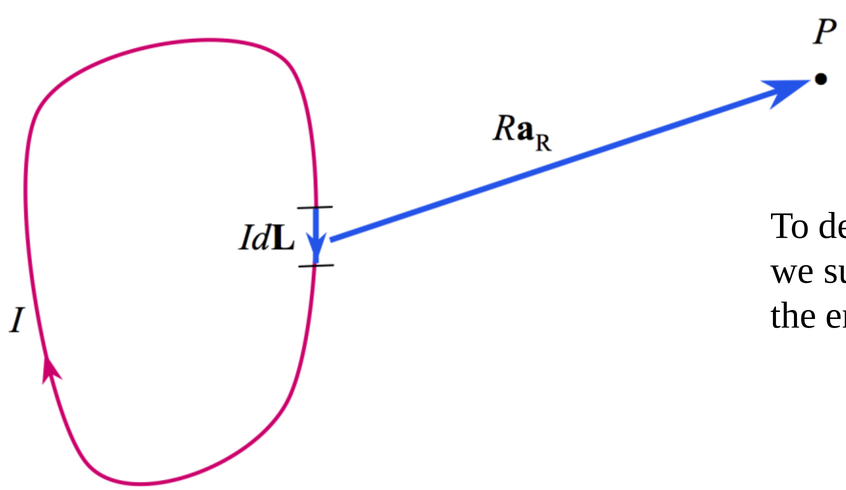


Note the similarity to Coulomb's Law, in which a point charge of magnitude dQ_1 at Point 1 would generate electric field at Point 2 given by:

$$\boxed{d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi \epsilon_0 R_{12}^2}}$$

06 HL Passing: L3	Apply known magnetic field laws to quantify different magnetic fields due to direct current	Solve the equation and verify the formulated field with the symmetry found earlier	Mathematical exposition of the required field in vector form	Identify the symmetry in the problem and apply Biot-Savart law or Ampere's law for the required expression of the field	Draw graphical representation of the problem at hand	Cannot draw the physical picture of the problem at hand

Magnetic Field Arising From a Circulating Current



At point P, the magnetic field associated with the differential current element Id**L** is

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

To determine the total field arising from the closed circuit path, we sum the contributions from the current elements that make up the entire loop, or

$$\boxed{\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}}$$

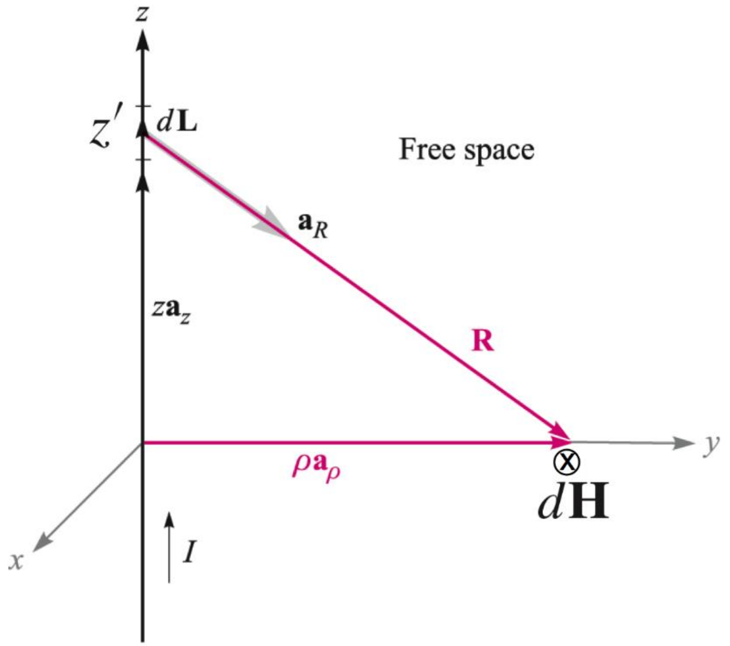
The contribution to the field at P from any portion of the current will be just the above integral evalated over just that portion.

Example of the Biot-Savart Law

In this example, we evaluate the magnetic field intensity on the y axis (equivalently in the xy plane) arising from a filament current of infinite length in on the z axis.

Using the drawing, we identify:

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \rho \mathbf{a}_\rho - z' \mathbf{a}_z$$



and so.. $$\mathbf{a}_R = \frac{\rho \mathbf{a}_\rho - z' \mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

so that:

$$d\mathbf{H} = \frac{I\, d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\, dz' \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi (\rho^2 + z'^2)^{3/2}}$$

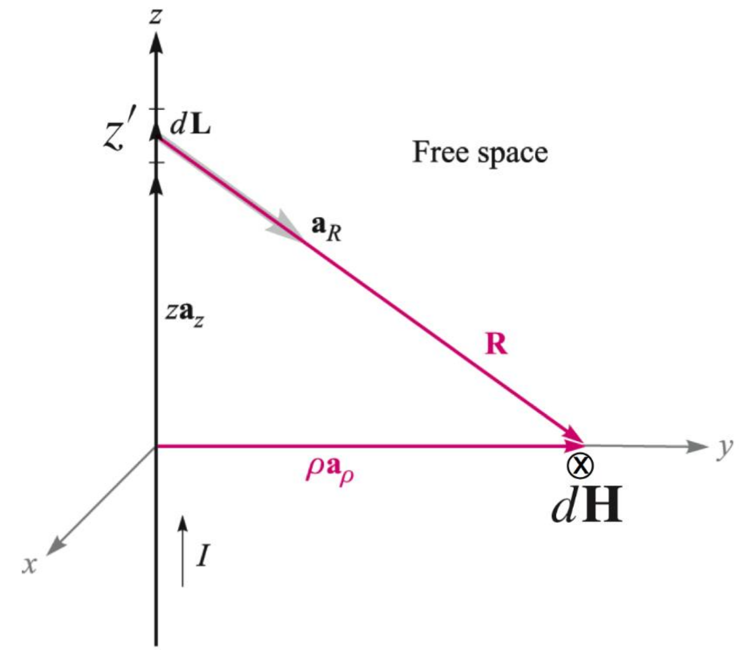
Example: continued

We now have: $$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Integrate this over the entire wire:

$$\mathbf{H} = \int_{-\infty}^{\infty} \frac{I\,dz'\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z'\mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi}\int_{-\infty}^{\infty} \frac{\rho\,dz'\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$



..after carrying out the cross product

Example: concluded

Evaluating the integral:

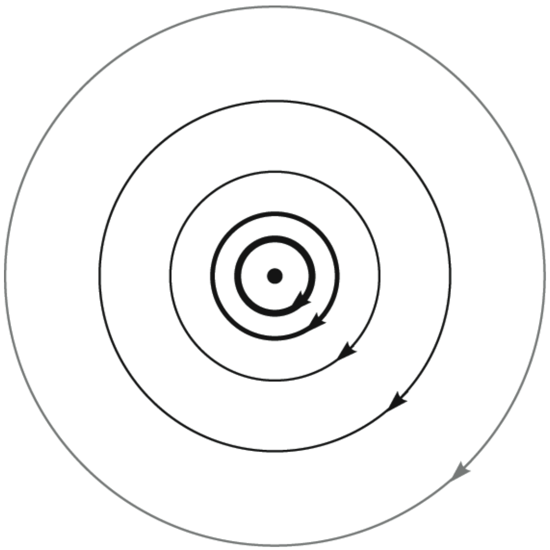
we have:

$$\mathbf{H} = \frac{I}{4\pi}\int_{-\infty}^{\infty}\frac{\rho\, dz'\,\mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I\rho\mathbf{a}_\phi}{4\pi}\left.\frac{z'}{\rho^2\sqrt{\rho^2 + z'^{\,2}}}\right|_{-\infty}^{\infty}$$

finally:

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi}$$



Current is into the page.
Magnetic field streamlines
are concentric circles, whose magnitudes
decrease as the inverse distance from the z axis

Field Arising from a Finite Current Segment

In this case, the field is to be found in the xy plane at Point 2.
The Biot-Savart integral is taken over the wire length:

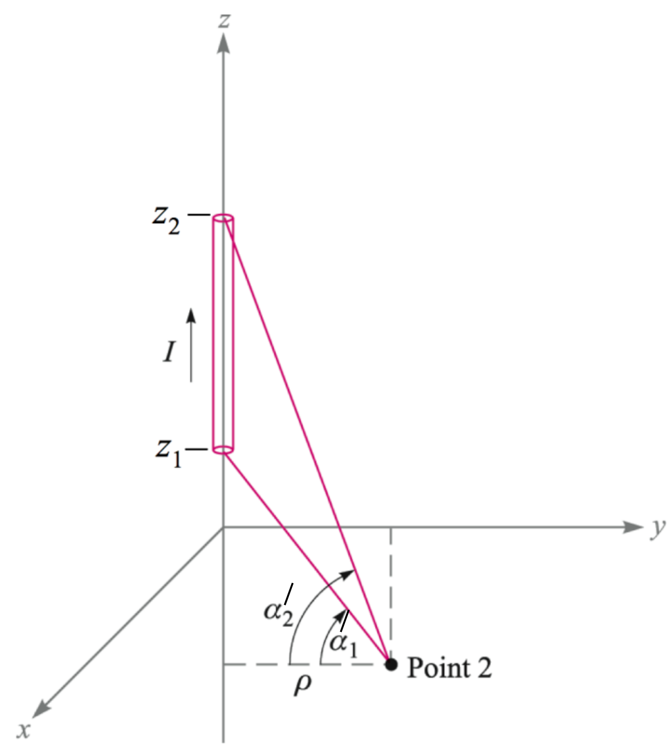
$$\mathbf{H} = \int_{z_1}^{z_2} \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

$A = \pi r^2$

$$= \int_{\rho \tan \alpha_1}^{\rho \tan \alpha_2} \frac{Idz\mathbf{a}_z \times (\rho \mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi(\rho^2 + z^2)^{3/2}}$$

..after a few additional steps (see Problem 7.8), we find:

$$\boxed{\mathbf{H} = \frac{I}{4\pi\rho}(\sin \alpha_2' - \sin \alpha_1')\mathbf{a}_\phi}$$

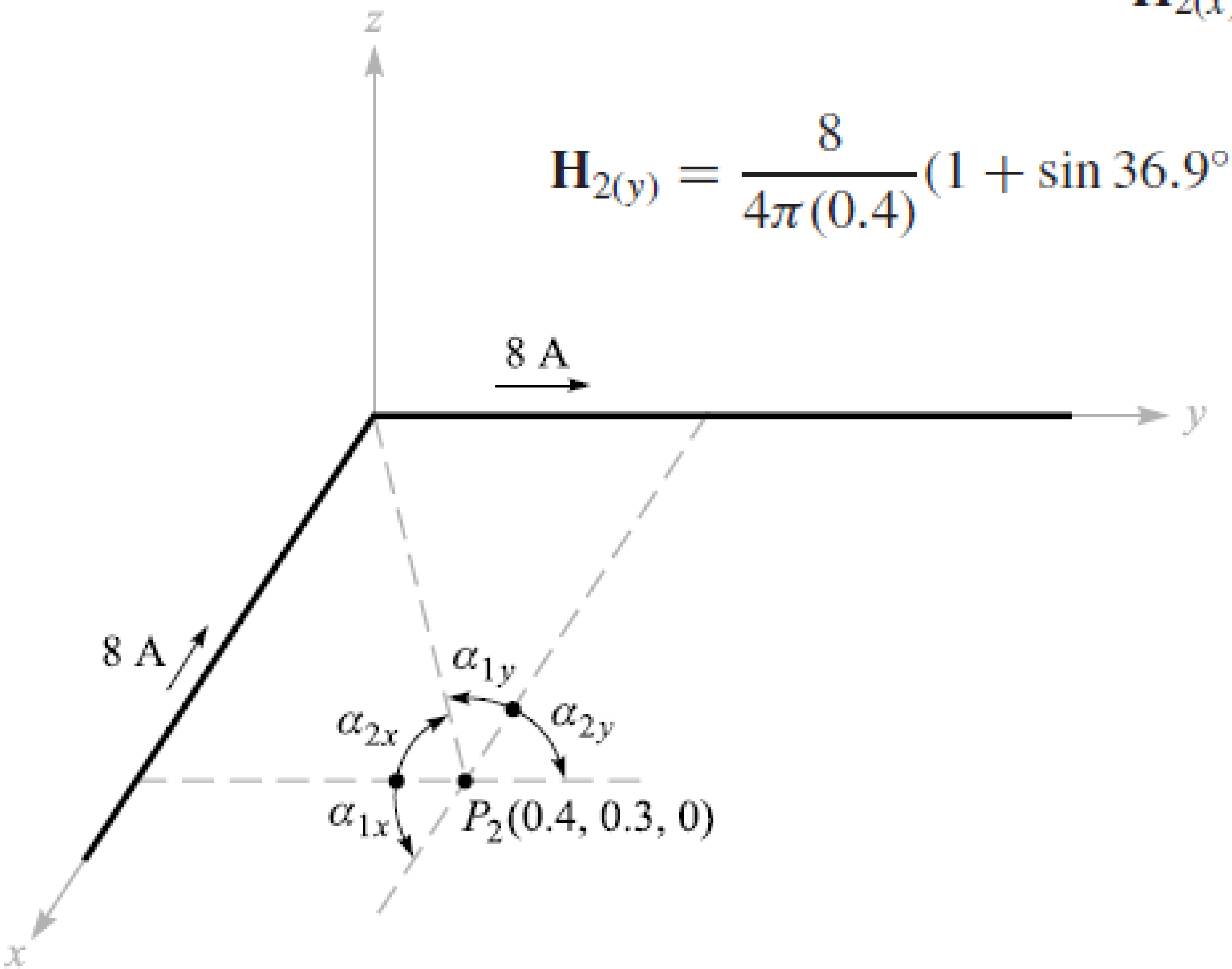


α' = angle of elevation as discussed in notes

Example 7.1

$$\mathbf{H}_{2(x)} = -\frac{12}{\pi}\mathbf{a}_z \text{ A/m}$$

$$\mathbf{H}_{2(y)} = \frac{8}{4\pi(0.4)}(1 + \sin 36.9^\circ)(-\mathbf{a}_z) = -\frac{8}{\pi}\mathbf{a}_z \text{ A/m}$$



D7.1. Given the following values for P_1, P_2, and $I_1 \Delta L_1$, calculate $\Delta \mathbf{H}_2$: (*a*) $P_1(0, 0, 2)$, $P_2(4, 2, 0)$, $2\pi \mathbf{a}_z \,\mu\text{A}\cdot\text{m}$; (*b*) $P_1(0, 2, 0)$, $P_2(4, 2, 3)$, $2\pi \mathbf{a}_z \,\mu\text{A}\cdot\text{m}$; (*c*) $P_1(1, 2, 3)$, $P_2(-3, -1, 2)$, $2\pi(-\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)\,\mu\text{A}\cdot\text{m}$.

Ans. $-8.51\mathbf{a}_x + 17.01\mathbf{a}_y$ nA/m; $16\mathbf{a}_y$ nA/m; $18.9\mathbf{a}_x - 33.9\mathbf{a}_y + 26.4\mathbf{a}_z$ nA/m

D7.2. A current filament carrying 15 A in the $\mathbf{a}_z$ direction lies along the entire z axis. Find $\mathbf{H}$ in rectangular coordinates at: (*a*) $P_A(\sqrt{20}, 0, 4)$; (*b*) $P_B(2, -4, 4)$.

Ans. $0.534\mathbf{a}_y$ A/m; $0.477\mathbf{a}_x + 0.239\mathbf{a}_y$ A/m

Another Example Biot-Savart LAW: Magnetic Field from a Current Loop

Consider a circular current loop of radius a in the x-y plane, which carries steady current I. We wish to find the magnetic field strength anywhere on the z axis.

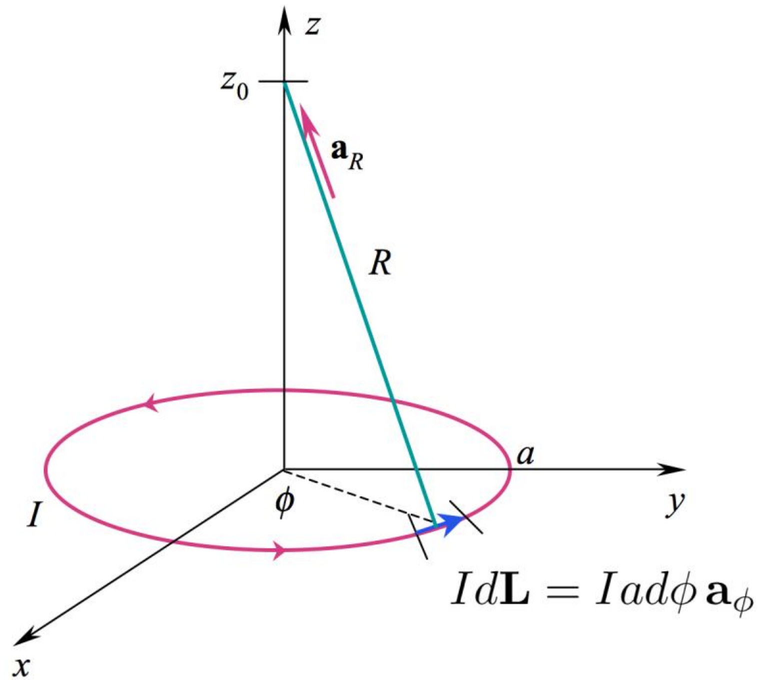
We will use the Biot-Savart Law:

$$\mathbf{H} = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

where: $I d\mathbf{L} = I a d\phi \, \mathbf{a}_\phi$

$$R = \sqrt{a^2 + z_0^2}$$

$$\mathbf{a}_R = \frac{z_0 \, \mathbf{a}_z - a \, \mathbf{a}_\rho}{\sqrt{a^2 + z_0^2}}$$



Cylindrical Co-ordinates

	$\boldsymbol{a}_\rho$	$\boldsymbol{a}_\varphi$	$\boldsymbol{a}_z$
$\boldsymbol{a}_x \cdot$	$\cos(\varphi)$	$-\sin(\varphi)$	0
$\boldsymbol{a}_y \cdot$	$\sin(\varphi)$	$\cos(\varphi)$	0
$\boldsymbol{a}_z \cdot$	0	0	1

Substituting the previous expressions, the Biot-Savart Law becomes:

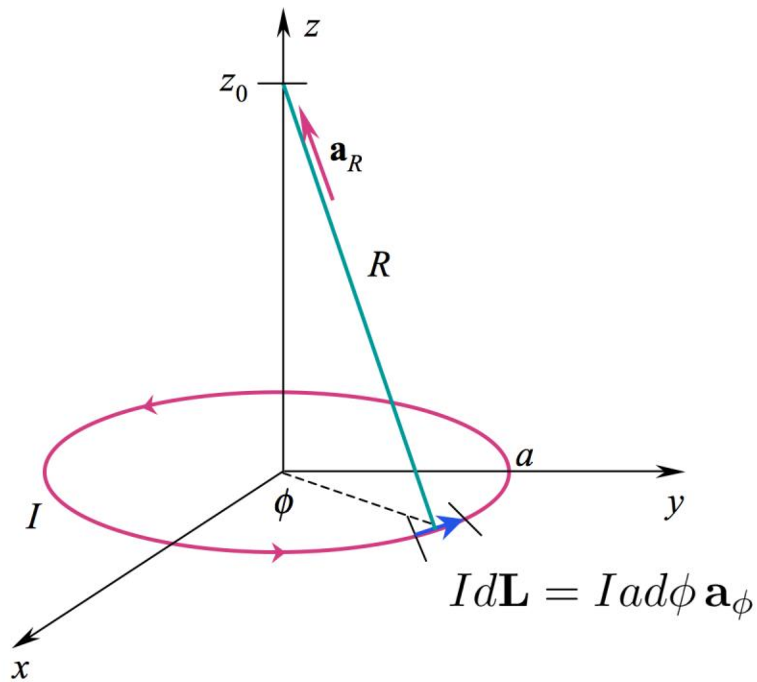
$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,\mathbf{a}_\phi \times (z_0\,\mathbf{a}_z - a\,\mathbf{a}_\rho)}{4\pi(a^2 + z_0^2)^{3/2}}$$

carry out the cross products to find:

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,(z_0\,\mathbf{a}_\rho + a\,\mathbf{a}_z)}{4\pi(a^2 + z_0^2)^{3/2}}$$

but we must include the angle dependence in the radial unit vector:

$$\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$$



with this substitution, the radial component will integrate to zero, meaning that all radial components will cancel on the z axis.

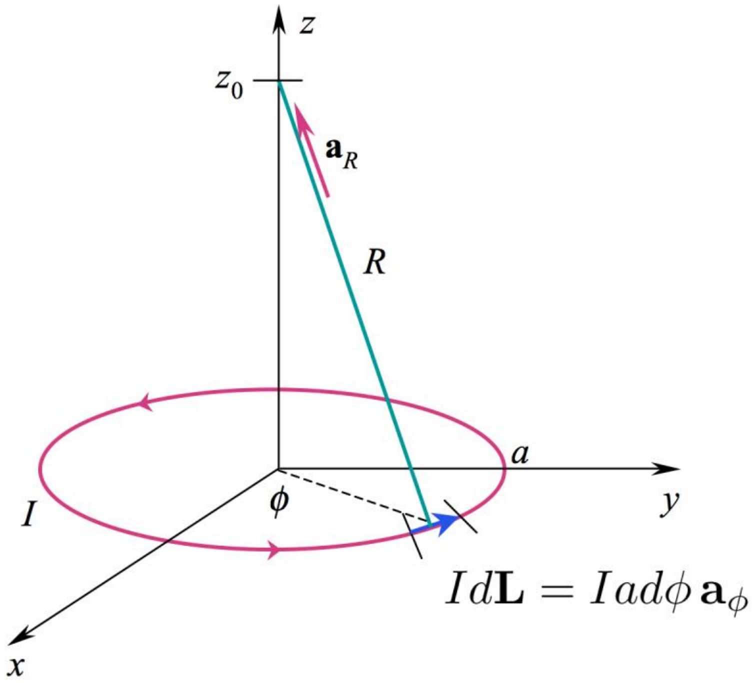
Magnetic Moment

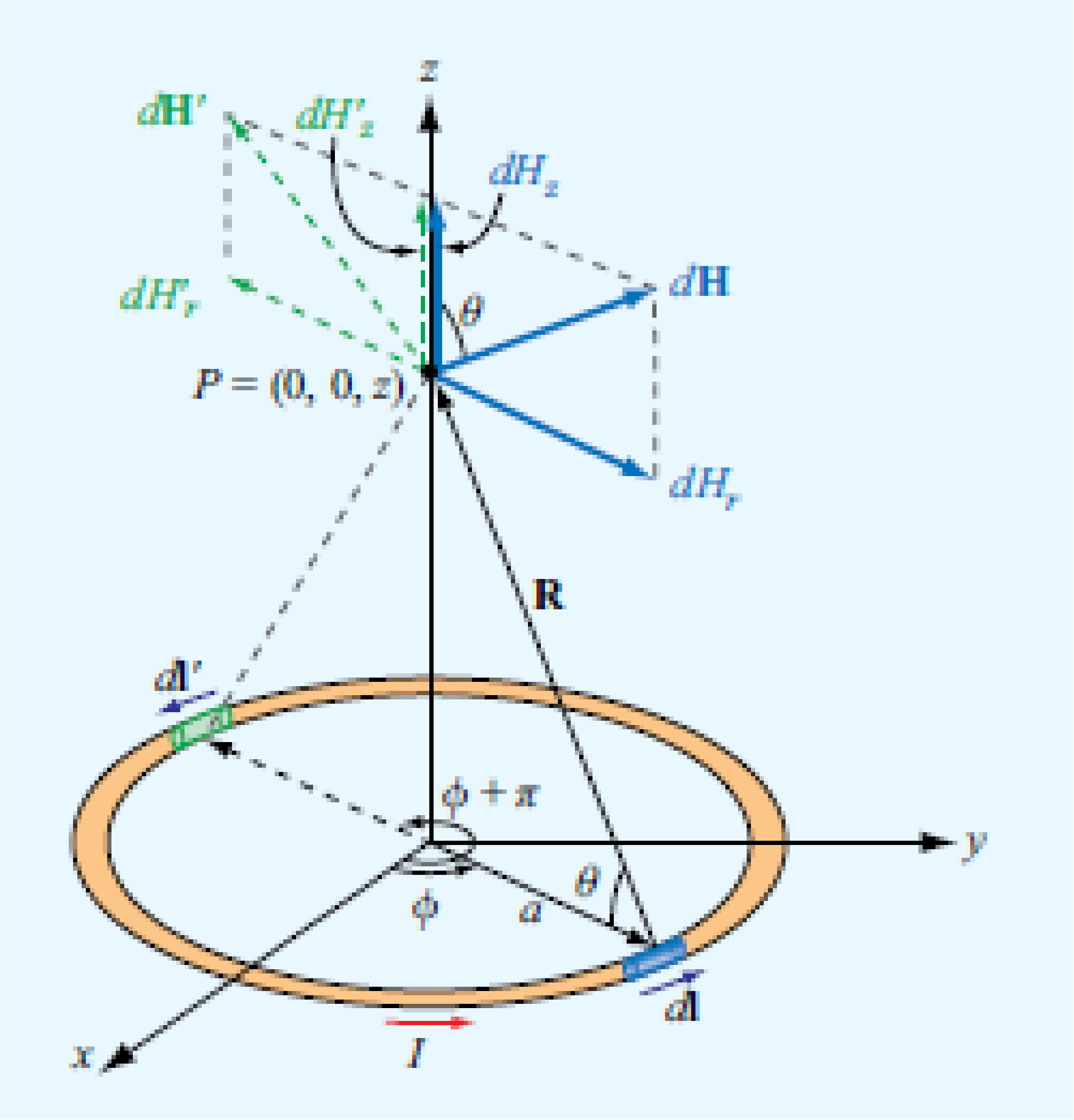
Now, only the z component remains, and the integral evaluates easily:

$$\boxed{\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}}$$

Note the form of the numerator: the product of the current and the loop area. We define this as the magnetic moment:

$$\mathbf{m} = I(\pi a^2)\,\mathbf{a}_z$$



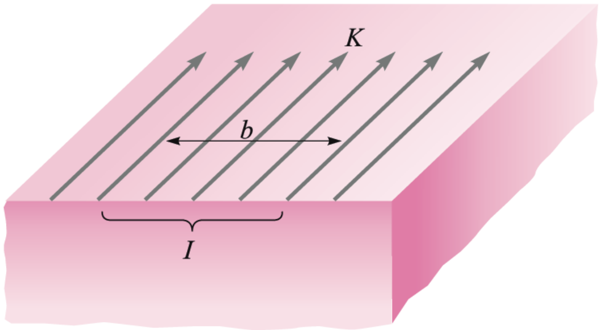


z
dH'
dH'z
dHz
dH
dH'r
θ
P = (0, 0, z)
dHr
R
dl'
φ + π
y
θ
φ
a
dl
x
I

Two- and Three-Dimensional Currents

On a surface that carries uniform surface current density **K** [A/m], the current within width b is

$$I = Kb$$



..and so the differential current quantity that appears in the Biot-Savart law becomes:

$$I\,d\mathbf{L} = \mathbf{K}\,dS$$

The magnetic field arising from a current sheet is thus found from the two-dimensional form of the Biot-Savart law:

$$\boxed{\mathbf{H} = \int_S \frac{\mathbf{K} \times \mathbf{a}_R\,dS}{4\pi R^2}}$$

In a similar way, a **volume current** will be made up of three-dimensional current elements, and so the Biot-Savart law for this case becomes:

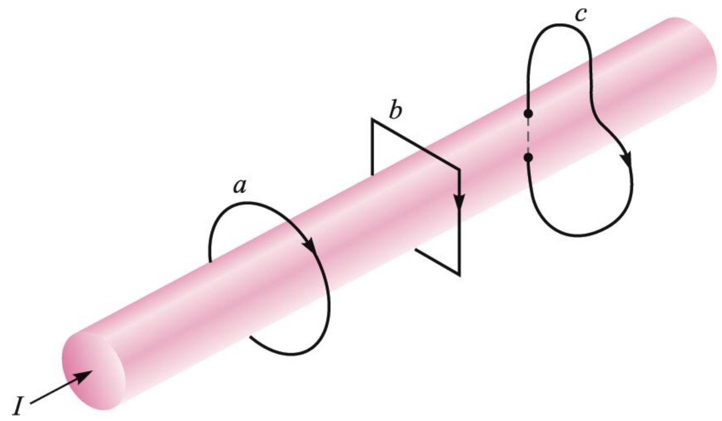
$$\boxed{\mathbf{H} = \int_{\mathrm{vol}} \frac{\mathbf{J} \times \mathbf{a}_R\,dv}{4\pi R^2}}$$

Ampere’s Circuital Law

Ampere' s Circuital Law

Ampere' s Circuital Law states that the line integral of **H** about any closed path is exactly equal to the direct current enclosed by that path.

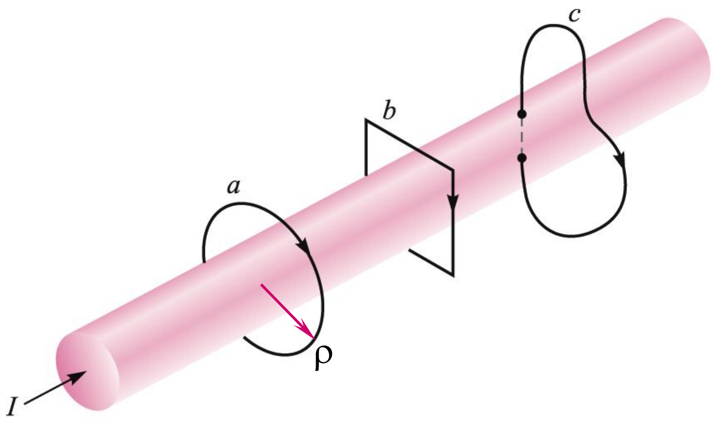
$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$



In the figure at right, the integral of **H** about closed paths a and b gives the total current I, while the integral over path c gives only that portion of the current that lies within c

Ampere's Law Applied to a Long Wire

Ampere's Law Applied to a Long Wire



Symmetry suggests that **H** will be circular, constant-valued at constant radius, and centered on the current (z) axis.

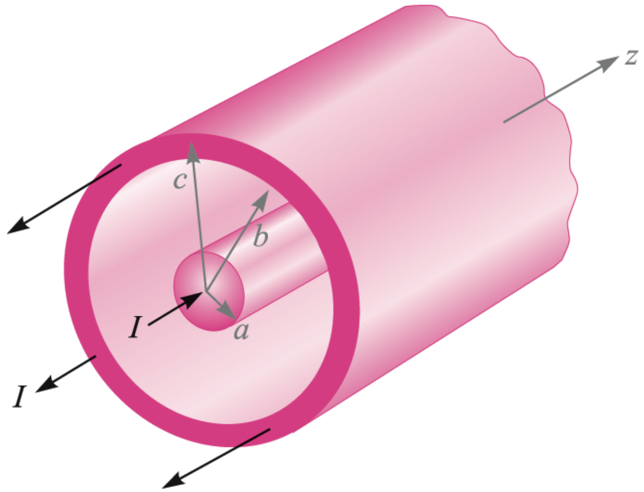
Choosing path a, and integrating **H** around the circle of radius ρ gives the enclosed current, I:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho \, d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi \rho = I$$

so that: $\boxed{H_\phi = \dfrac{I}{2\pi\rho}}$ as before.

Ampere's Law Applied to Coaxial Transmission Line

Coaxial Transmission Line



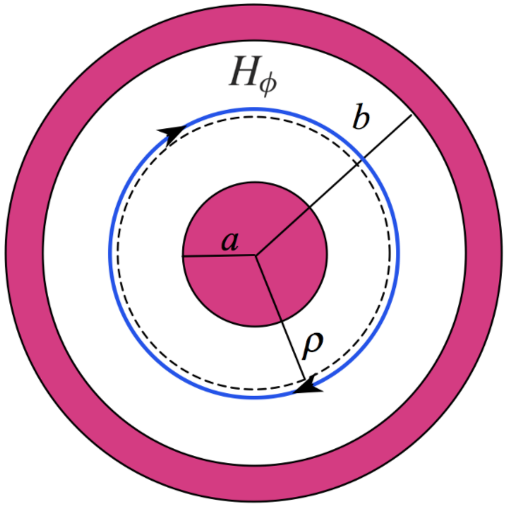
In the coax line, we have two concentric solid conductors that carry equal and opposite currents, I.

The line is assumed to be infinitely long, and the circular symmetry suggests that **H** will be entirely ϕ - directed, and will vary only with radius ρ.

Our objective is to find the magnetic field for all values of ρ

Field Between Conductors

$$H_\phi = \frac{I}{2\pi\rho} \qquad a < \rho < b$$

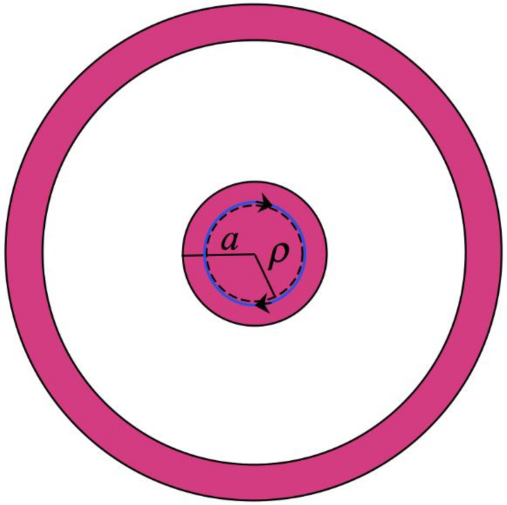


Field Within the Inner Conductor

With current uniformly distributed inside the conductors, the **H** can be assumed circular everywhere.

Inside the inner conductor, and at radius ρ, we again have:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi \ = H_\phi 2\pi \rho$$



But now, the current enclosed is $I_{\text{encl}} = I \dfrac{\rho^2}{a^2}$

so that $2\pi \rho H_\phi = I \dfrac{\rho^2}{a^2}$ or finally: $\boxed{H_\phi = \dfrac{I\rho}{2\pi a^2} \quad (\rho < a)}$

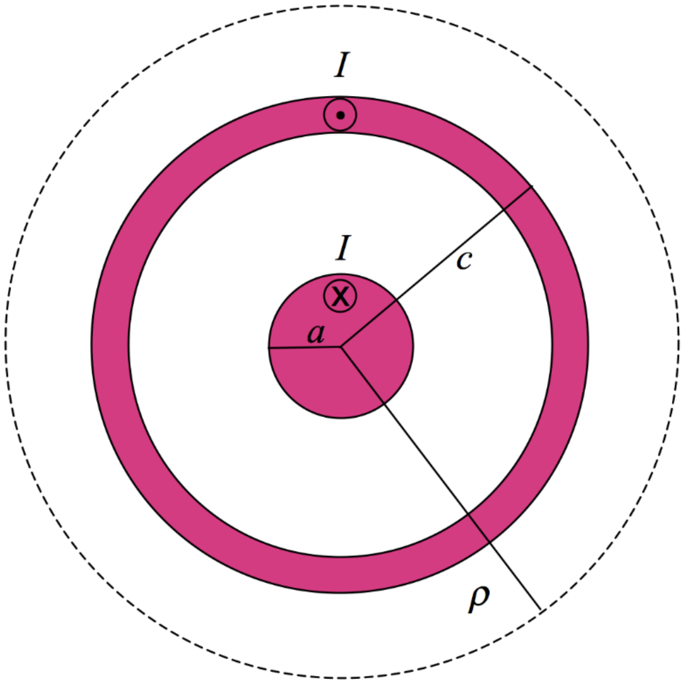
Field Outside Both Conducors

Outside the transmission line, where ρ> c,
no current is enclosed by the integration path,
and so

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0$$

As the current is uniformly distributed, and since we have circular symmetry, the field would have to be constant over the circular integration path, and so it must be true that:

$$\boxed{H_\phi = 0 \quad (\rho > c)}$$



Field Inside the Outer Conductor

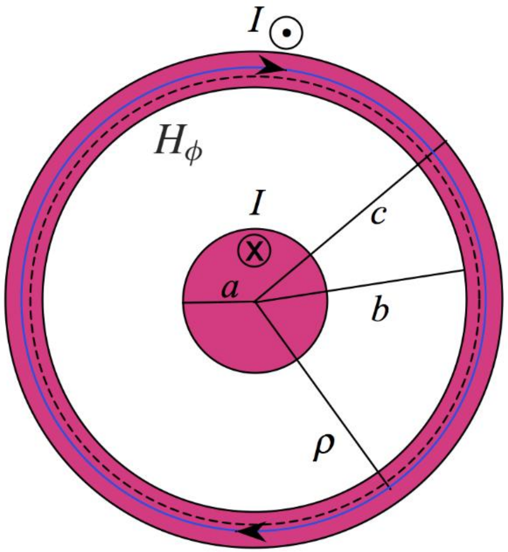
Inside the outer conductor, the enclosed current consists of that within the inner conductor plus that portion of the outer conductor current existing at radii less than ρ

Ampere's Circuital Law becomes

$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

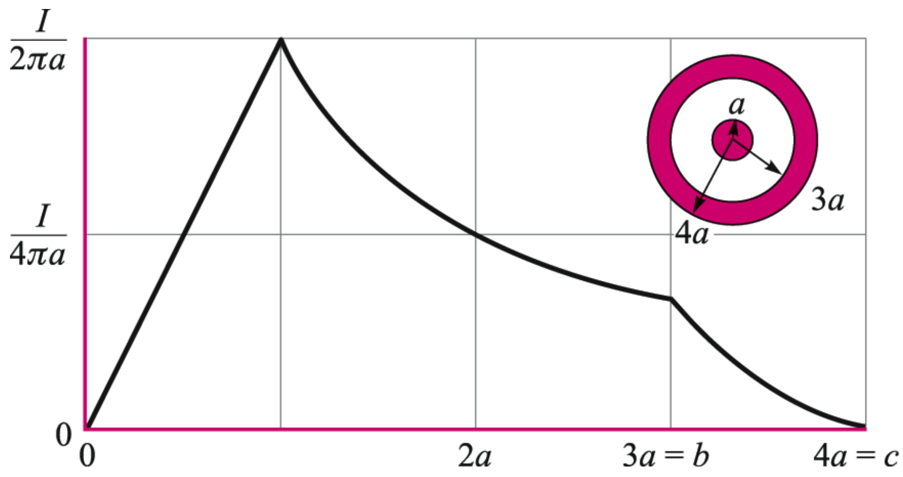
..and so finally:

$$\boxed{H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \quad (b < \rho < c)}$$



Magnetic Field Strength as a Function of Radius in the Coax Line

Combining the previous results, and assigning dimensions as shown in the inset below, we find:



Magnetic field due to a Current Sheet

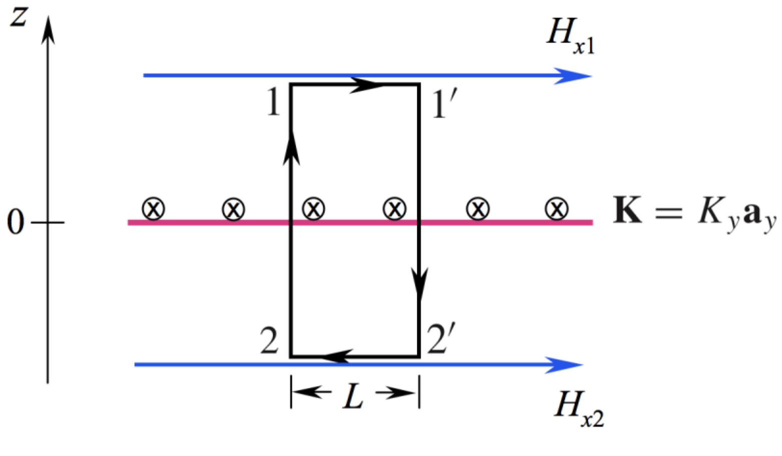
Magnetic Field Arising from a Current Sheet

For a uniform plane current in the y direction, we expect an x-directed **H** field from symmetry.

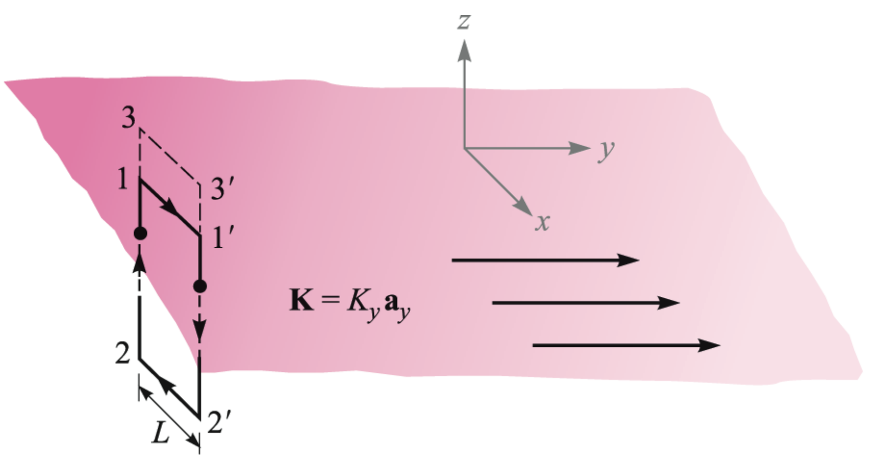
Applying Ampere's circuital law to the path $1\text{-}1'\text{-}2'\text{-}2\text{-}1$ we find:

$$H_{x1}L + H_{x2}(-L) = K_y L \quad \text{or} \quad H_{x1} - H_{x2} = K_y$$

In other words, the magnetic field is discontinuous across the current sheet by the magnitude of the surface current density.



edge view

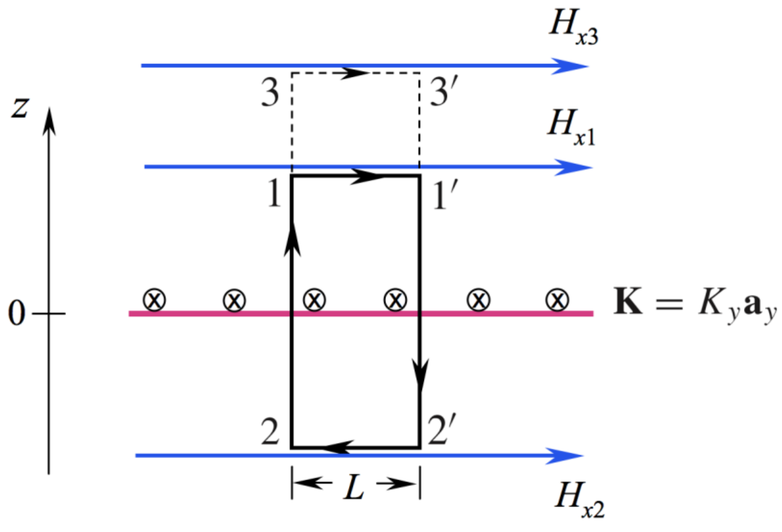


Magnetic Field Arising from a Current Sheet

If instead, the upper path is elevated to the line between 3 and $3'$, the same current is enclosed and we would have

$$H_{x3} - H_{x2} = K_y$$ from which we conclude that $$H_{x3} = H_{x1}$$

so the field is constant in each region (above and below the current plane)



edge view

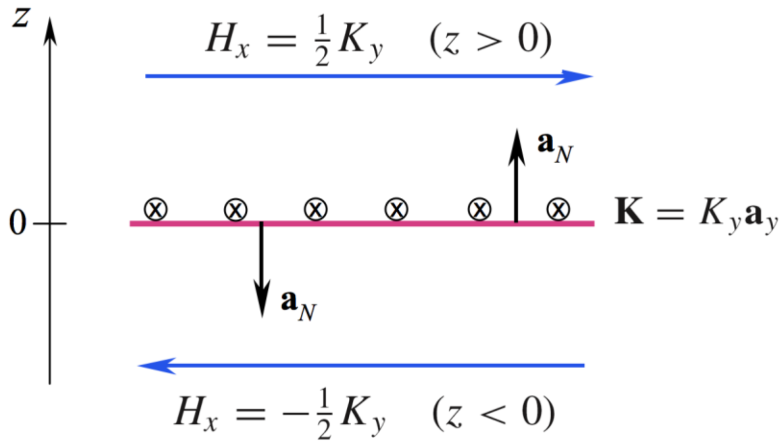
By symmetry, the field above the sheet must be the same in magnitude as the field below the sheet. Therefore, we may state that

$$\boxed{H_x = \frac{1}{2}K_y \quad (z > 0)}$$

and $$H_x = -\frac{1}{2}K_y \quad (z < 0)$$

Magnetic Field Arising from a Current Sheet

The actual field configuration is shown below, in which magnetic field above the current sheet is equal in magnitude, but in the direction opposite to the field below the sheet.



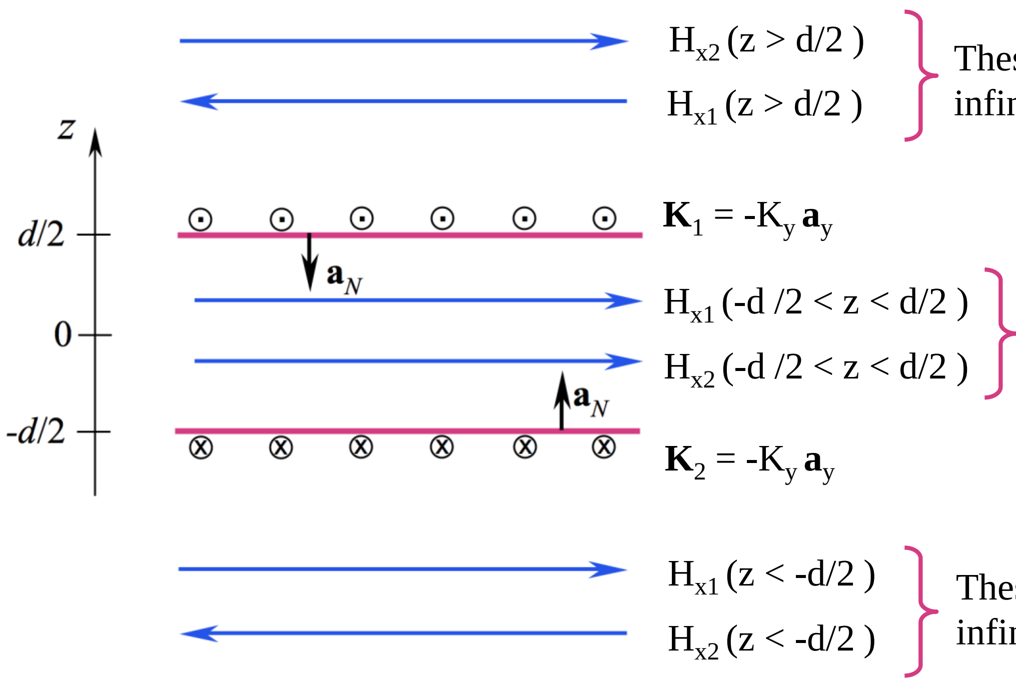
The field in either region is found by the cross product:

$$\boxed{\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N}$$

where $\mathbf{a}_N$ is the unit vector that is normal to the current sheet, and that points into the region in which the magnetic field is to be evaluated.

Magnetic Field Arising from Two Current Sheets

Here are two parallel currents, equal and opposite, as you would find in a parallel-plate transmission line. If the sheets are much wider than their spacing, then the magnetic field will be contained in the region between plates, and will be nearly zero outside.



These fields cancel for current sheets of infinite width.

These fields are equal and add to give

$$\boxed{\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (-d/2 < z < d/2\ )}$$

where $\mathbf{K}$ is either $\mathbf{K}_1$ or $\mathbf{K}_2$

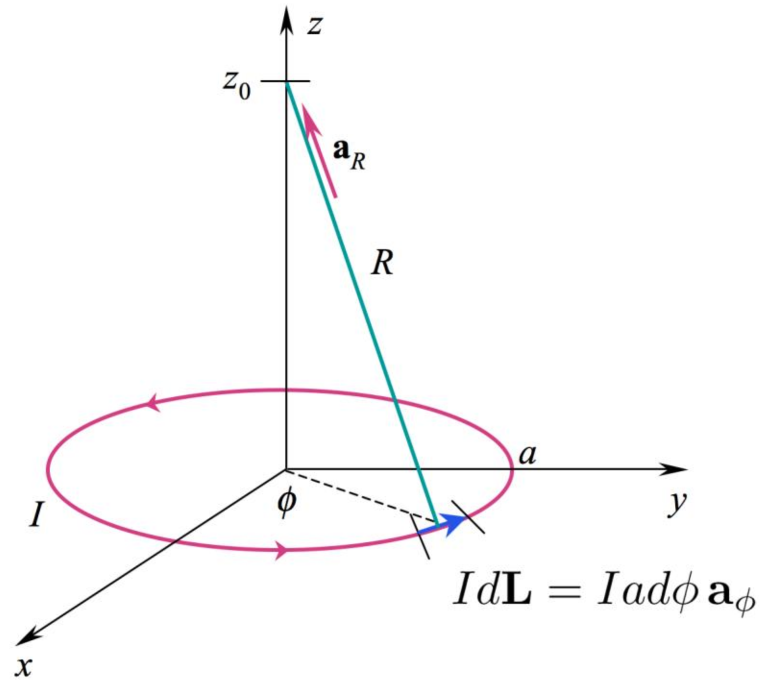
These fields cancel for current sheets of infinite width.

Magnetic field due to a Solenoid

Current Loop Field

Using the Biot-Savart Law, we previously found the magnetic field on the z axis from a circular current loop:

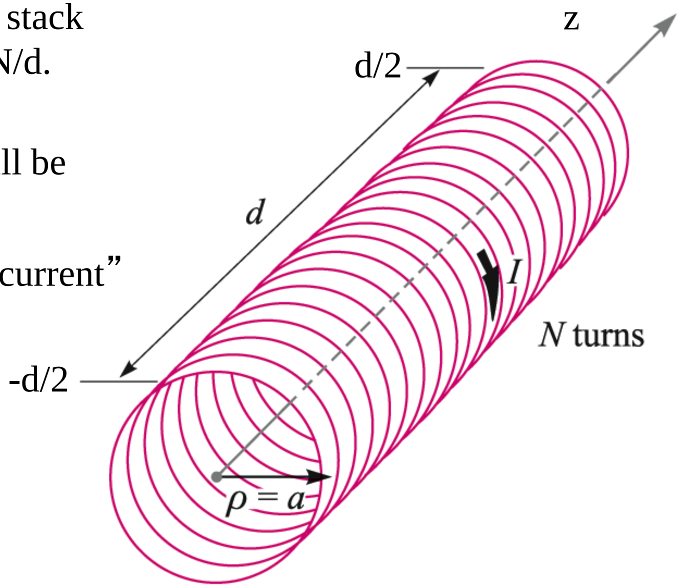
$$\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$$



We will now use this result as a building block to construct the magnetic field on the axis of a solenoid -- formed by a stack of identical current loops, centered on the z axis.

On-Axis Field Within a Solenoid

We consider the single current loop field as a differential contribution to the total field from a stack of N closely-spaced loops, each of which carries current I. The length of the stack (solenoid) is d, so therefore the density of turns will be N/d.



Now the current in the turns within a differential length, dz, will be

$$\boxed{dI = \frac{N}{d} I dz}$$

We consider this as our differential "loop current"

so that the previous result for **H** from a single loop:

$$\mathbf{H} = \frac{I(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z_0^2)^{3/2}}$$

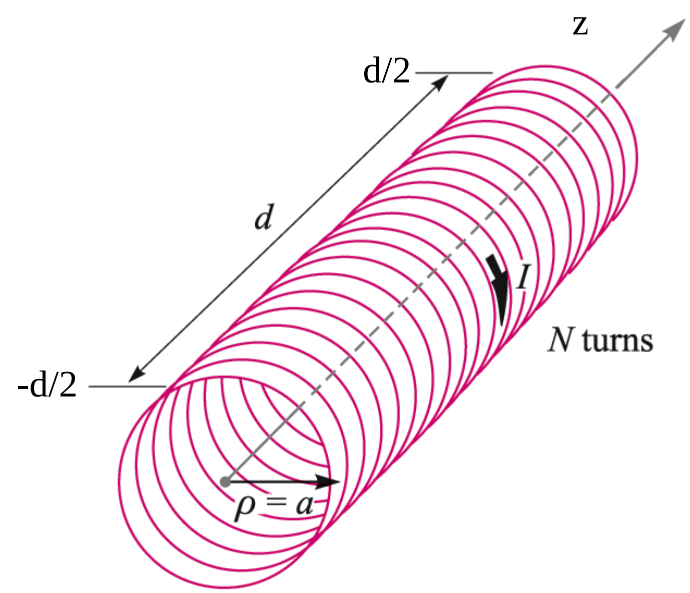
now becomes:

$$d\mathbf{H} = \frac{(N/d)Idz(\pi a^2)\mathbf{a}_z}{2\pi(a^2 + z^2)^{3/2}}$$

in which z is measured from the center of the coil, where we wish to evaluate the field.

Solenoid Field, Continued

The total field on the z axis at z = 0 will be the sum of the field contributions from all turns in the coil -- or the integral of d**H** over the length of the solenoid.



$$\mathbf{H} = \int d\mathbf{H} = \int_{-d/2}^{d/2} \frac{(N/d)I\,dz(\pi a^2)\mathbf{a}_z}{2\pi(a^2+z^2)^{3/2}}$$

$$= \frac{NIa^2}{2d}\,\mathbf{a}_z \int_{-d/2}^{d/2} \frac{dz}{(a^2+z^2)^{3/2}}$$

$$= \frac{NIa^2}{2d}\,\mathbf{a}_z \frac{d}{a^2\sqrt{a^2+(d/2)^2}} = \frac{NI\,\mathbf{a}_z}{2\sqrt{a^2+(d/2)^2}}$$

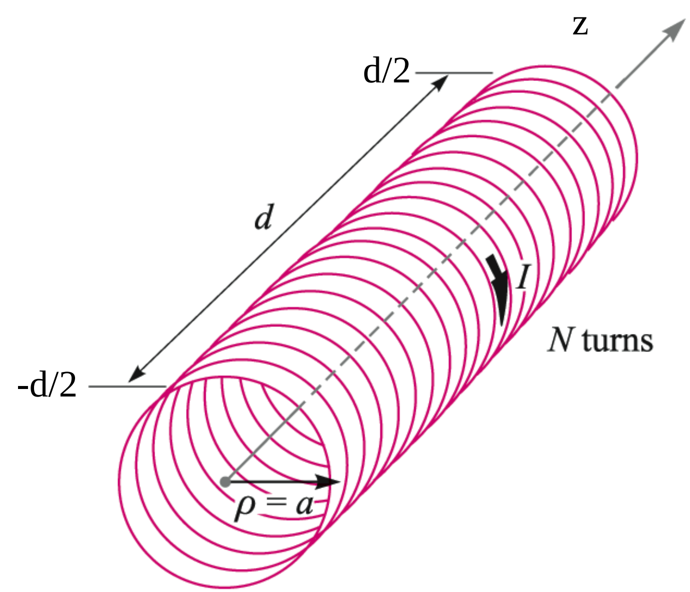
Approximation for Long Solenoids

We now have the on-axis field at the solenoid midpoint (z = 0):

$$\mathbf{H} = \frac{NI\,\mathbf{a}_z}{2\sqrt{a^2 + (d/2)^2}}$$

Note that for long solenoids, for which $d >> a$ the result simplifies to:

$$\boxed{\mathbf{H} \doteq \frac{NI}{d}\mathbf{a}_z \quad (d >> a)}$$

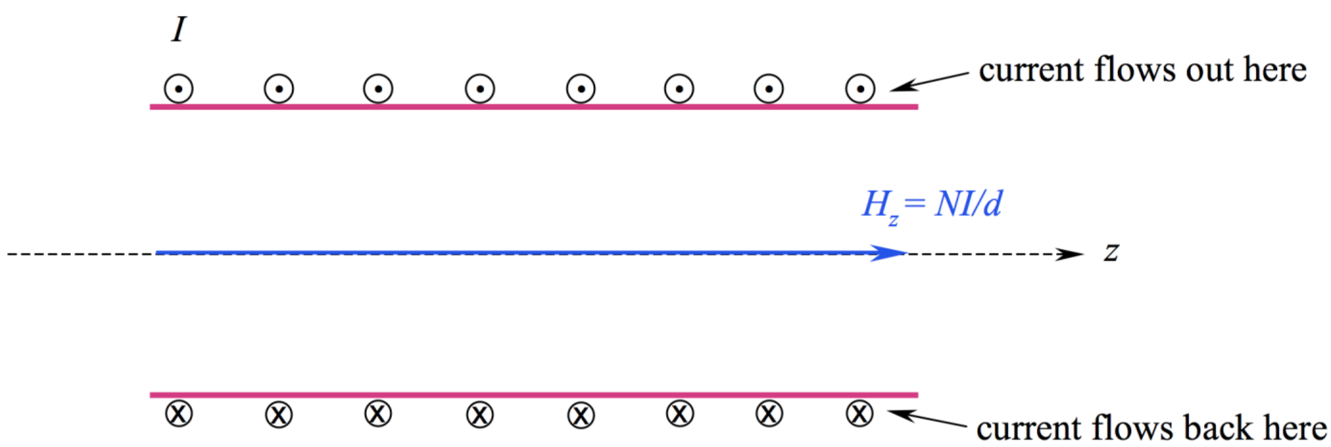


This result is valid at all on-axis positions deep within long coils -- at distances from each end of several radii.

Solenoid Field – Ampere's Law

To find the field within a solenoid, we apply Ampere's Circuital Law in the following way:

The illustration below shows the solenoid cross-section, from a lengthwise cut through the z axis. Current in the windings flows in and out of the screen in the circular current path. Each turn carries current I. The magnetic field along the z axis is NI/d as we found earlier.

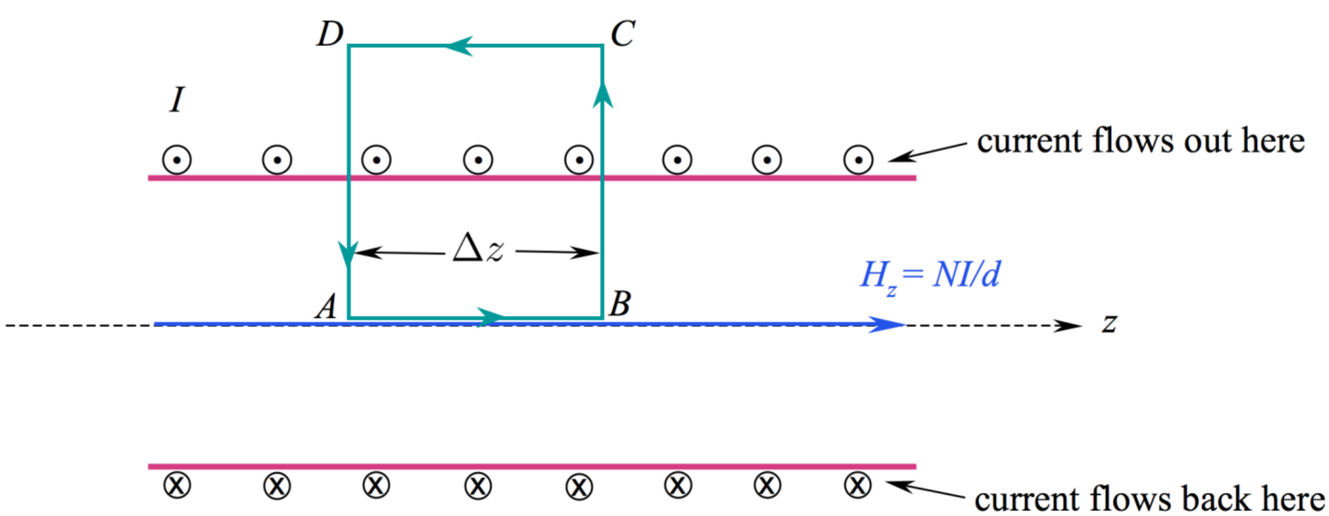


Application of Ampere' s Law

Applying Ampere' s Law to the rectangular path shown below leads to the following:

$$\oint \mathbf{H} \cdot d\mathbf{L} \;=\; \int_A^B H_z\, dz + \int_B^C H_\rho\, d\rho + \int_C^D H_{z,out}\, dz + \int_D^A H_\rho\, d\rho \;=\; I_{encl} \;=\; \frac{NI}{d}\Delta z$$

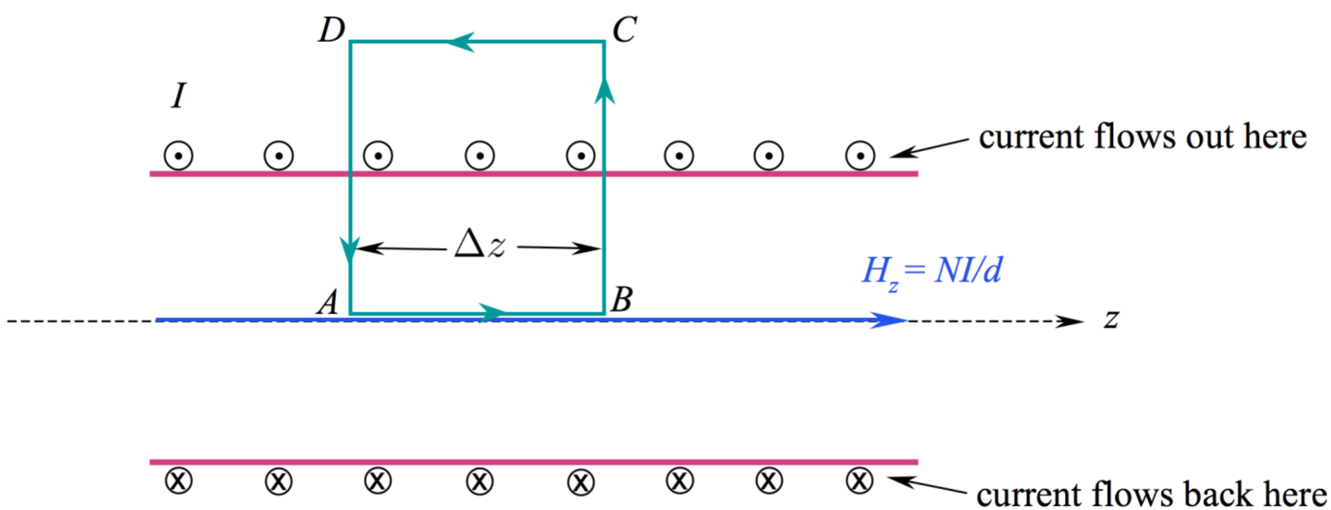
Where allowance is made for the existence of a radial H component, H_ρ



Radial Path Segments

The radial integrals will now cancel, because they are oppositely-directed, and because in the long coil, H_ρ is not expected to differ between the two radial path segments.

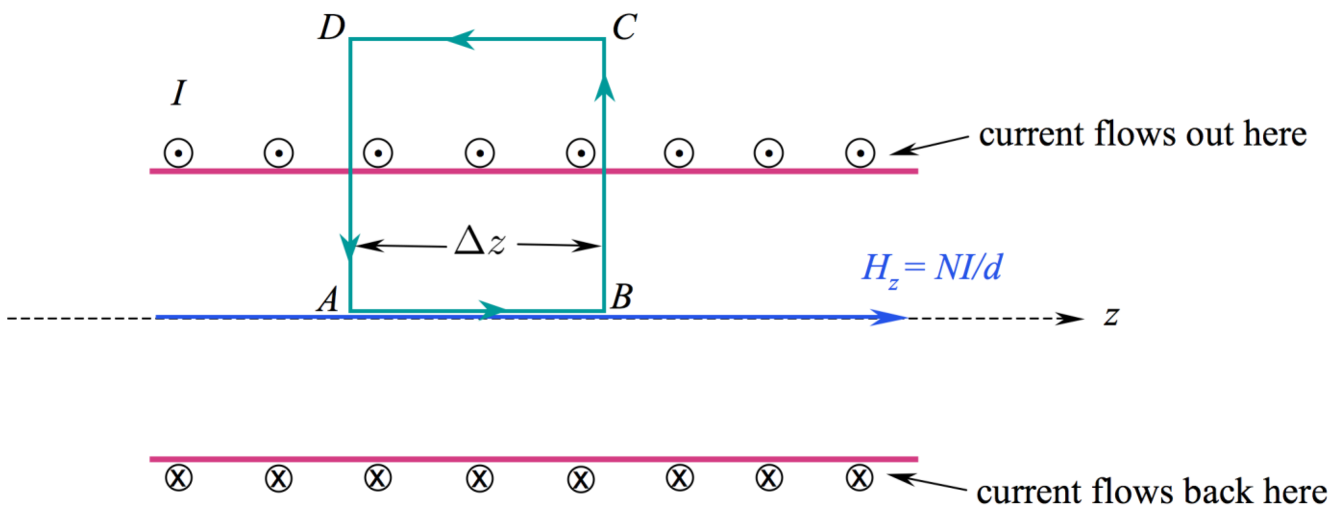
$$\oint \mathbf{H}\cdot d\mathbf{L} \;=\; \int_A^B H_z\,dz + \cancel{\int_B^C H_\rho\,d\rho} + \int_C^D H_{z,out}\,dz + \cancel{\int_D^A H_\rho\,d\rho} \;=\; I_{encl} \;=\; \frac{NI}{d}\Delta z$$



Completing the Evaluation

What is left now are the two z integrations, the first of which we can evaluate as shown. Since this first integral result is equal to the enclosed current, it must follow that the second integral -- and the outside magnetic field -- are zero.

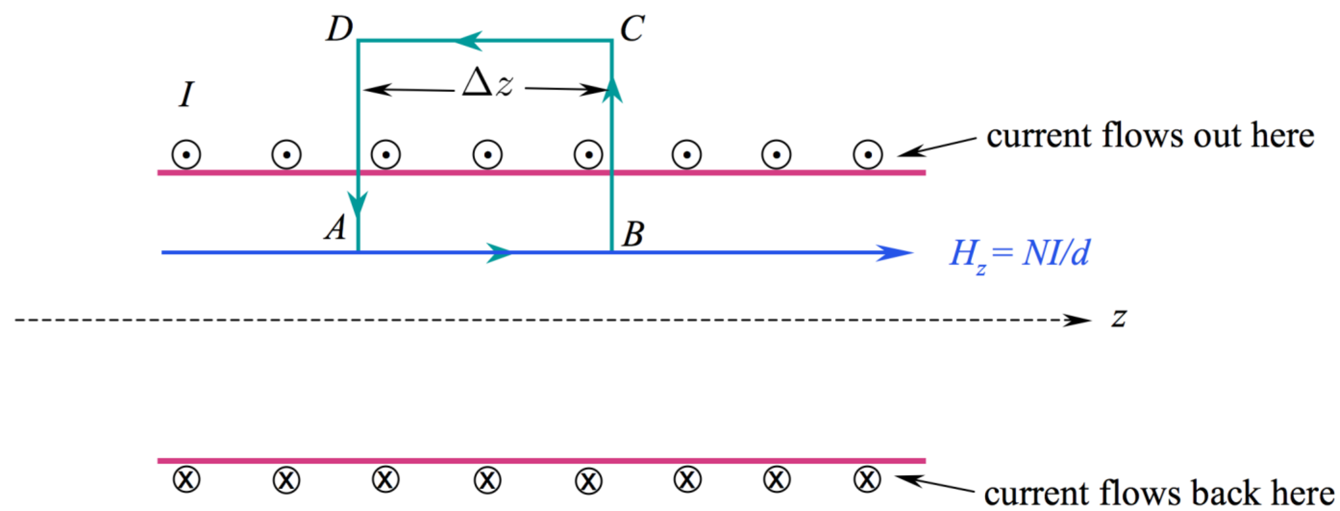
$$\oint \mathbf{H}\cdot d\mathbf{L} \;=\; \underbrace{\int_A^B H_z\,dz}_{(NI/d)\Delta z} + \cancel{\int_C^D H_{z,out}\,dz} \;=\; I_{encl} \;=\; \frac{NI}{d}\Delta z$$



Finding the Off-Axis Field

The situation does not change if the lower z-directed path is raised above the z axis. The vertical paths still cancel, and the outside field is still zero. The field along the path A to B is therefore NI/d as before.

$$\oint \mathbf{H}\cdot d\mathbf{L} = \underbrace{\int_A^B H_z\,dz}_{(NI/d)\Delta z} + \cancel{\int_C^D H_{z,out}\,dz} = I_{encl} = \frac{NI}{d}\Delta z$$



Conclusion: The magnetic field within a long solenoid is approximately constant throughout the coil cross-section, and is $H_z = NI/d$.

D7.3. Express the value of $\mathbf{H}$ in rectangular components at $P(0, 0.2, 0)$ in the field of: (*a*) a current filament, 2.5 A in the $\mathbf{a}_z$ direction at $x = 0.1$, $y = 0.3$; (*b*) a coax, centered on the z axis, with $a = 0.3$, $b = 0.5$, $c = 0.6$, $I = 2.5$ A in the $\mathbf{a}_z$ direction in the center conductor; (*c*) three current sheets, $2.7\mathbf{a}_x$ A/m at $y = 0.1$, $-1.4\mathbf{a}_x$ A/m at $y = 0.15$, and $-1.3\mathbf{a}_x$ A/m at $y = 0.25$.

Ans. $1.989\mathbf{a}_x - 1.989\mathbf{a}_y$ A/m; $-0.884\mathbf{a}_x$ A/m; $1.300\mathbf{a}_z$ A/m

Plane $y = 1$ carries current $\mathbf{K} = 50\mathbf{a}_z$ mA/m. Find $\mathbf{H}$ at

(a) $(0, 0, 0)$

(b) $(1, 5, -3)$

Answer: (a) $25\mathbf{a}_x$ mA/m, (b) $-25\mathbf{a}_x$ mA/m.

07 LL Passing:L3	Define physical interpretation of curl and divergence with application of divergence and Stoke's theorems	Apply the relevant theorem and evaluate both sides	Able to solve one side of the theorem to find the unknown field	Select suitable limits of integration involved in the problem	Identify the differential path, surface, and volume prevalent in the problem with physical interpretation	Cannot identify the shape and relevant tool to solve the problem

Curl

Curl of a Vector Field

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N}$$

where ΔS_N is the planar area enclosed by the closed line integral.

The direction of N is taken using the right-hand convention: With fingers of the right hand oriented in the direction of the path integral, the thumb points in the direction of the normal (or curl).

Curl in Rectangular Coordinates

Assembling the results of the rectangular loop integration exercise, we find the vector field that comprises curl **H**:

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

An easy way to calculate this is to evaluate the following determinant:

$$\text{curl } \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

which we see is equivalent to the cross product of the del operator with the field:

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H}$$

Curl in Other Coordinate Systems

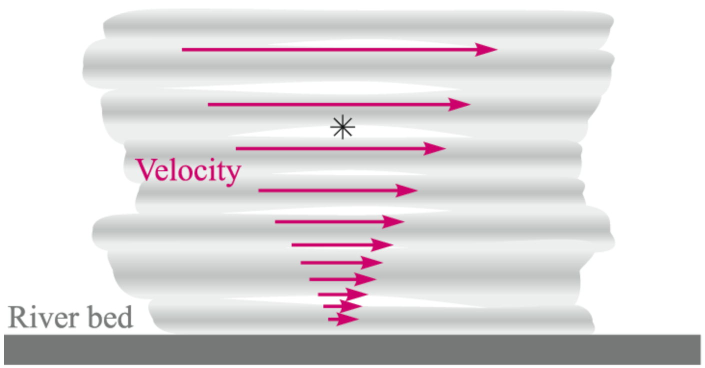
$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z \quad \text{(cylindrical)}$$

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi \quad \text{(spherical)}$$

Visualization of Curl

Consider placing a small “paddle wheel” in a flowing stream of water, as shown below. The wheel axis points into the screen, and the water velocity decreases with increasing depth.

The wheel will rotate clockwise, and give a curl component that points into the screen (right-hand rule).



Positioning the wheel at all three orthogonal orientations will yield measurements of all three components of the curl. Note that the curl is directed normal to both the field and the direction of its variation.

Another Maxwell Equation

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

This is Ampere's Circuital Law in point form.

We already know that for a static electric field:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

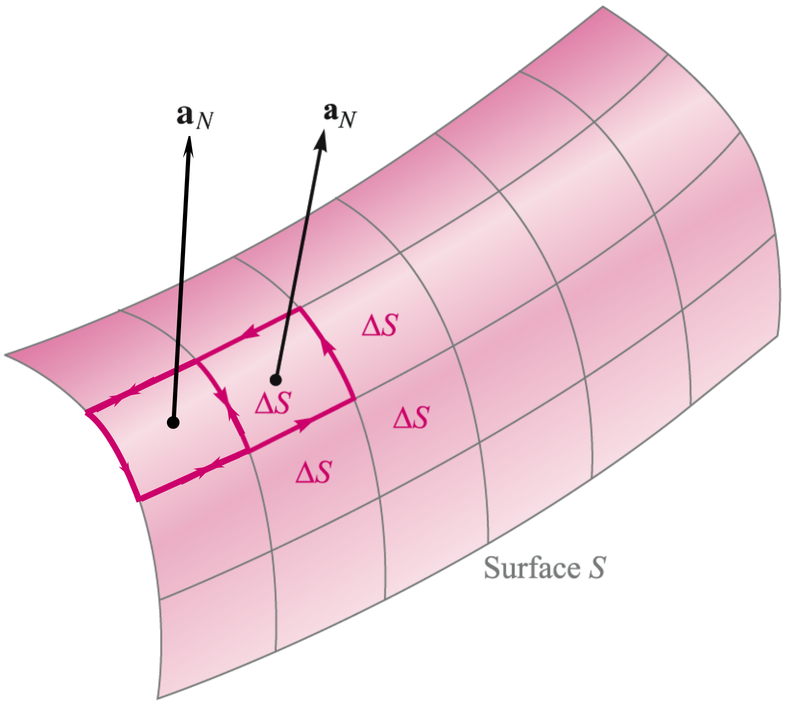
$$\boxed{\nabla \times \mathbf{E} = 0}$$

Therefore, a field is conservative if it has zero curl at all points over which the field is defined.

Stoke’s Theorem

Stokes' Theorem

$$\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$



Obtaining Ampere's Circuital Law in Integral Form, using Stokes' Theorem

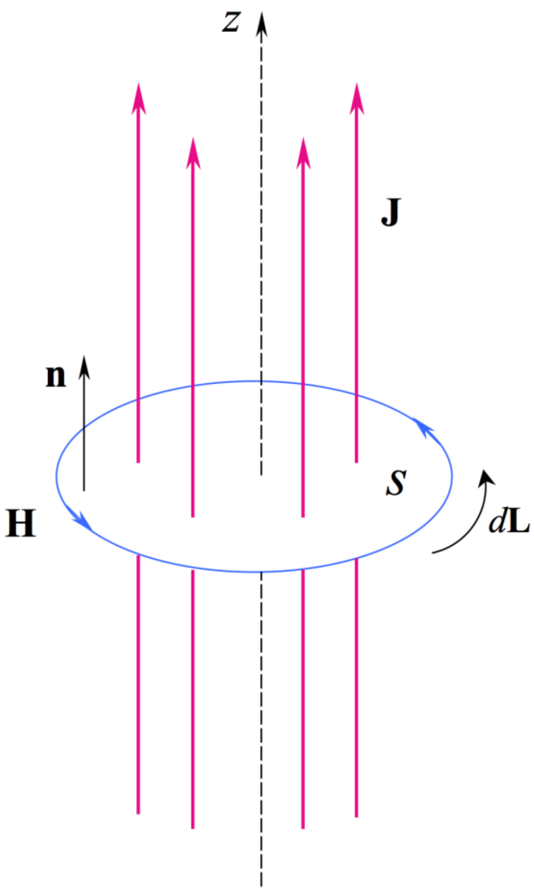
Begin with the point form of Ampere's Law for static fields:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Integrate both sides over surface S:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} = \oint \mathbf{H} \cdot d\mathbf{L}$$

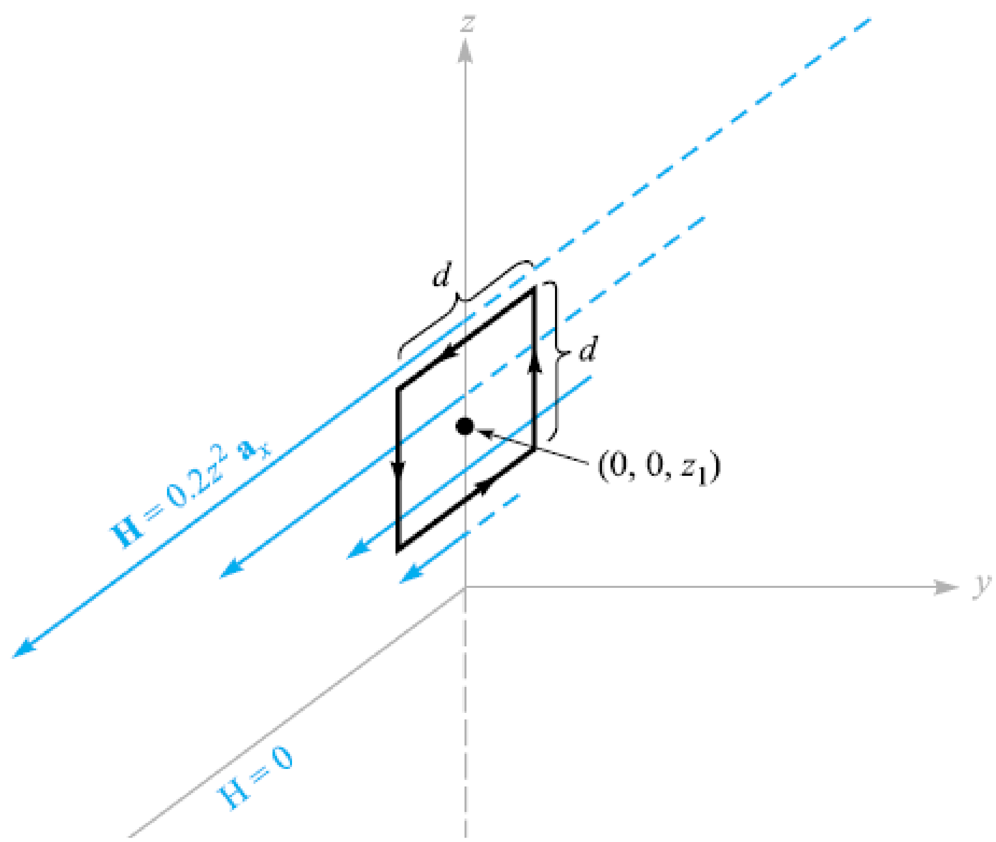
..in which the far right hand side is found from the left hand side using Stokes' Theorem. The closed path integral is taken around the perimeter of S. Again, note that we use the right-hand convention in choosing the direction of the path integral.



The center expression is just the net current through surface S, so we are left with the integral form of Ampere's Law:

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

As an example of the evaluation of curl $\mathbf{H}$ from the definition and of the evaluation of another line integral, suppose that $\mathbf{H} = 0.2z^2\mathbf{a}_x$ for $z > 0$, and $\mathbf{H} = 0$ elsewhere, as shown in Figure 7.15. Calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ about a square path with side d, centered at $(0, 0, z_1)$ in the $y = 0$ plane where $z_1 > d/2$.



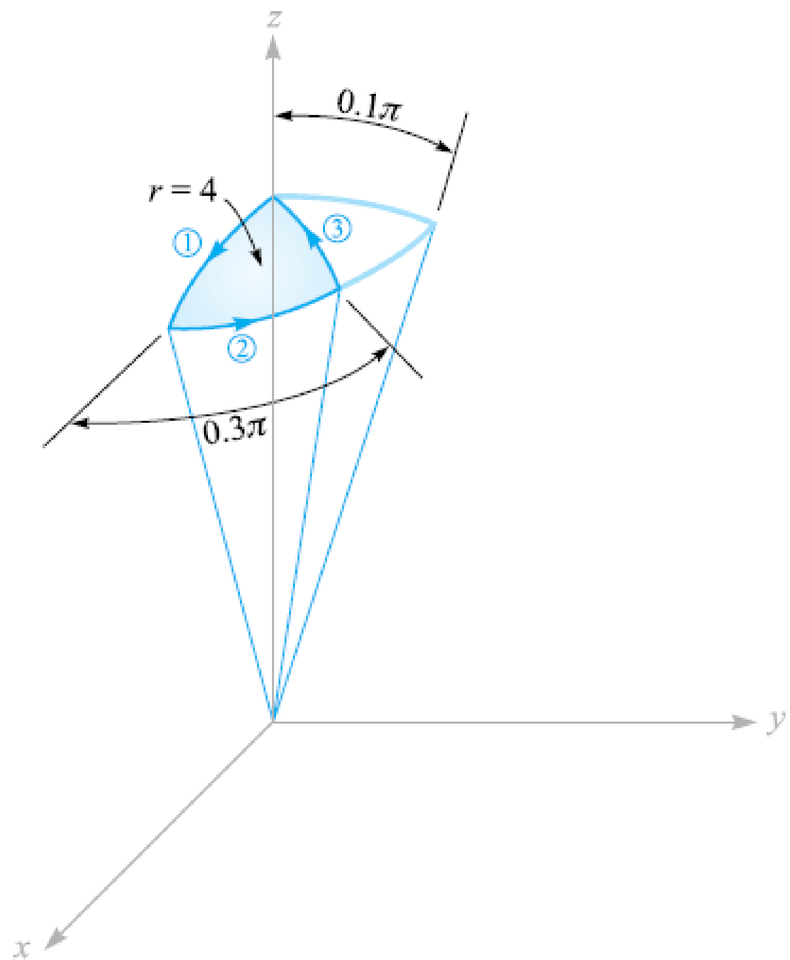
D7.4. (*a*) Evaluate the closed line integral of $\mathbf{H}$ about the rectangular path $P_1(2, 3, 4)$ to $P_2(4, 3, 4)$ to $P_3(4, 3, 1)$ to $P_4(2, 3, 1)$ to P_1, given $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$ A/m. (*b*) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to $(\nabla \times \mathbf{H})_y$. (*c*) Determine $(\nabla \times \mathbf{H})_y$ at the center of the area.

Ans. 354 A; 59 A/m^2; 57 A/m^2

D7.5. Calculate the value of the vector current density: (*a*) in rectangular coordinates at $P_A(2, 3, 4)$ if $\mathbf{H} = x^2z\mathbf{a}_y - y^2x\mathbf{a}_z$; (*b*) in cylindrical coordinates at $P_B(1.5, 90°, 0.5)$ if $\mathbf{H} = \dfrac{2}{\rho}(\cos 0.2\phi)\mathbf{a}_\rho$; (*c*) in spherical coordinates at $P_C(2, 30°, 20°)$ if $\mathbf{H} = \dfrac{1}{\sin\theta}\mathbf{a}_\theta$.

Ans. $-16\mathbf{a}_x + 9\mathbf{a}_y + 16\mathbf{a}_z$ A/m^2; $0.055\mathbf{a}_z$ A/m^2; $\mathbf{a}_\phi$ A/m^2

A numerical example may help to illustrate the geometry involved in Stokes' theorem. Consider the portion of a sphere shown in Figure 7.17. The surface is specified by $r = 4, 0 \leq \theta \leq 0.1\pi, 0 \leq \phi \leq 0.3\pi$, and the closed path forming its perimeter is composed of three circular arcs. We are given the field $\mathbf{H} = 6r \sin\phi \mathbf{a}_r + 18r \sin\theta \cos\phi \mathbf{a}_\phi$ and are asked to evaluate each side of Stokes' theorem.



D7.6. Evaluate both sides of Stokes' theorem for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \leq x \leq 5$, $-1 \leq y \leq 1$, $z = 0$. Let the positive direction of $d\mathbf{S}$ be $\mathbf{a}_z$.

Ans. -126 A; -126 A