Engineering Electromagnetics

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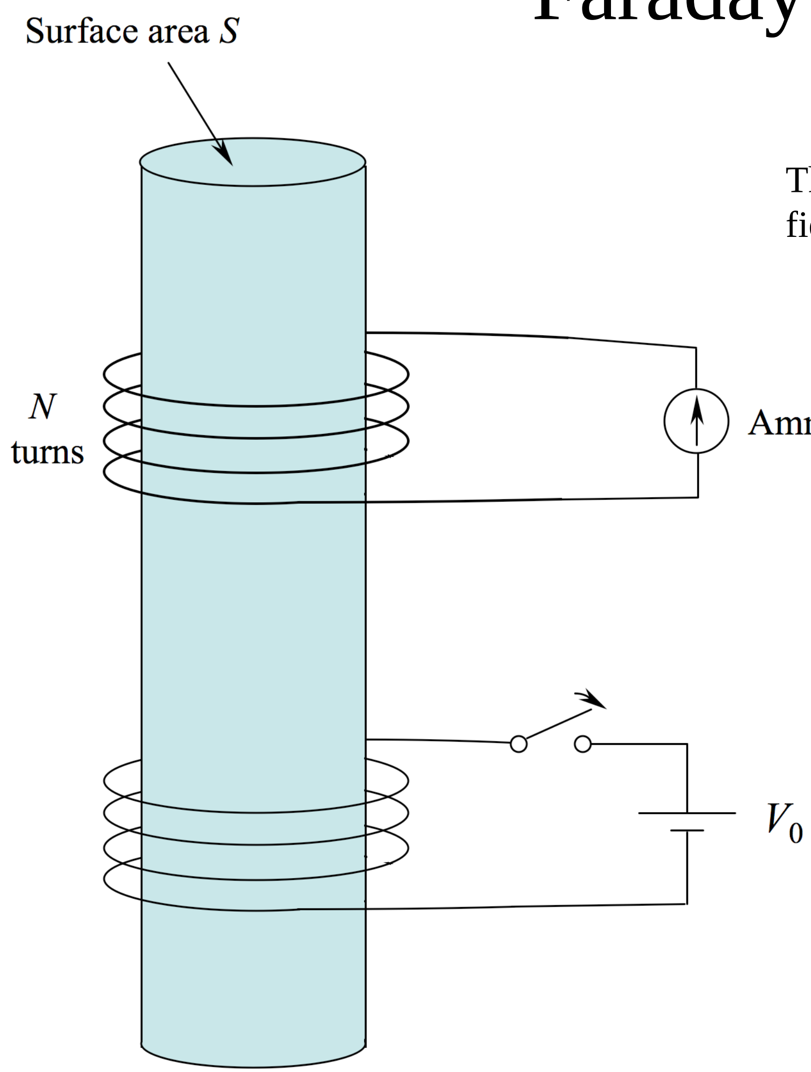
Chapter 9:

Time-Varying Fields

and Maxwell's Equations

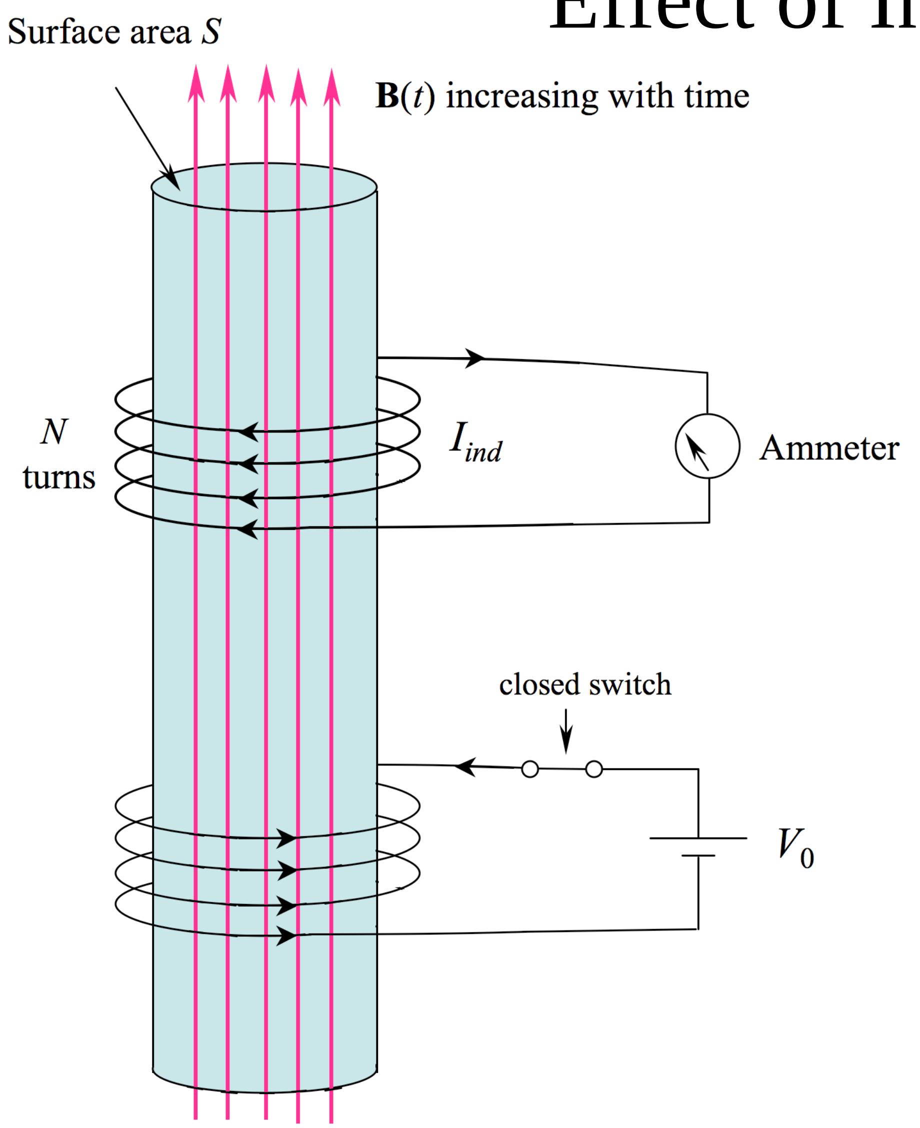
Faraday's Experiment



The question addressed is: If a current can generate a magnetic field, then can a magnetic field generate a current?

An experiment similar to that conducted to answer that question is shown here. Two sets of windings are placed on a shared iron core. In the lower set, a current is generated by closing the switch as shown. In the upper set, any induced current is registered by the ammeter.

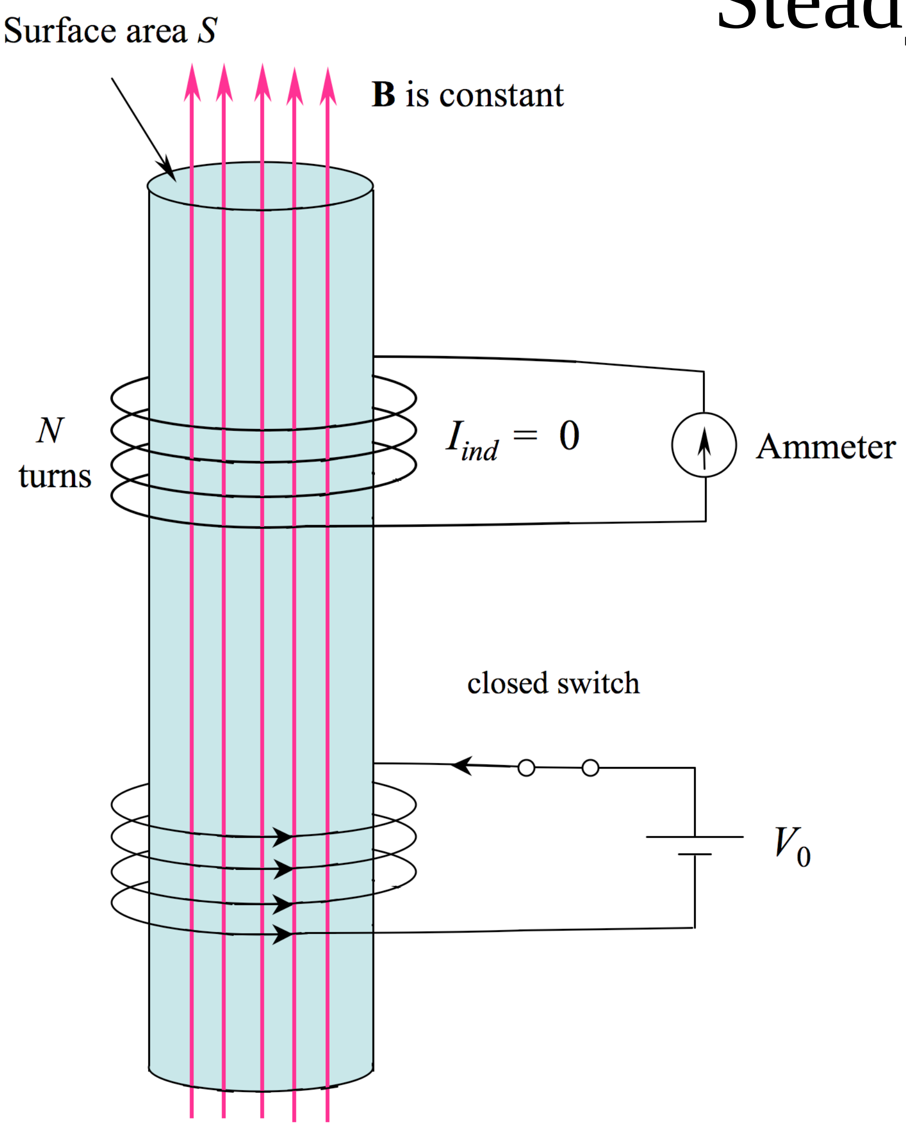
Effect of Increasing **B**



Closing the switch results in current that initially increases with time in the lower windings. Magnetic flux density **B** is generated, which now links the upper windings; these are found to carry current, I_{ind} . This current exists *as long as* **B** *increases with time*. Note the direction of the induced current and the position of the ammeter.

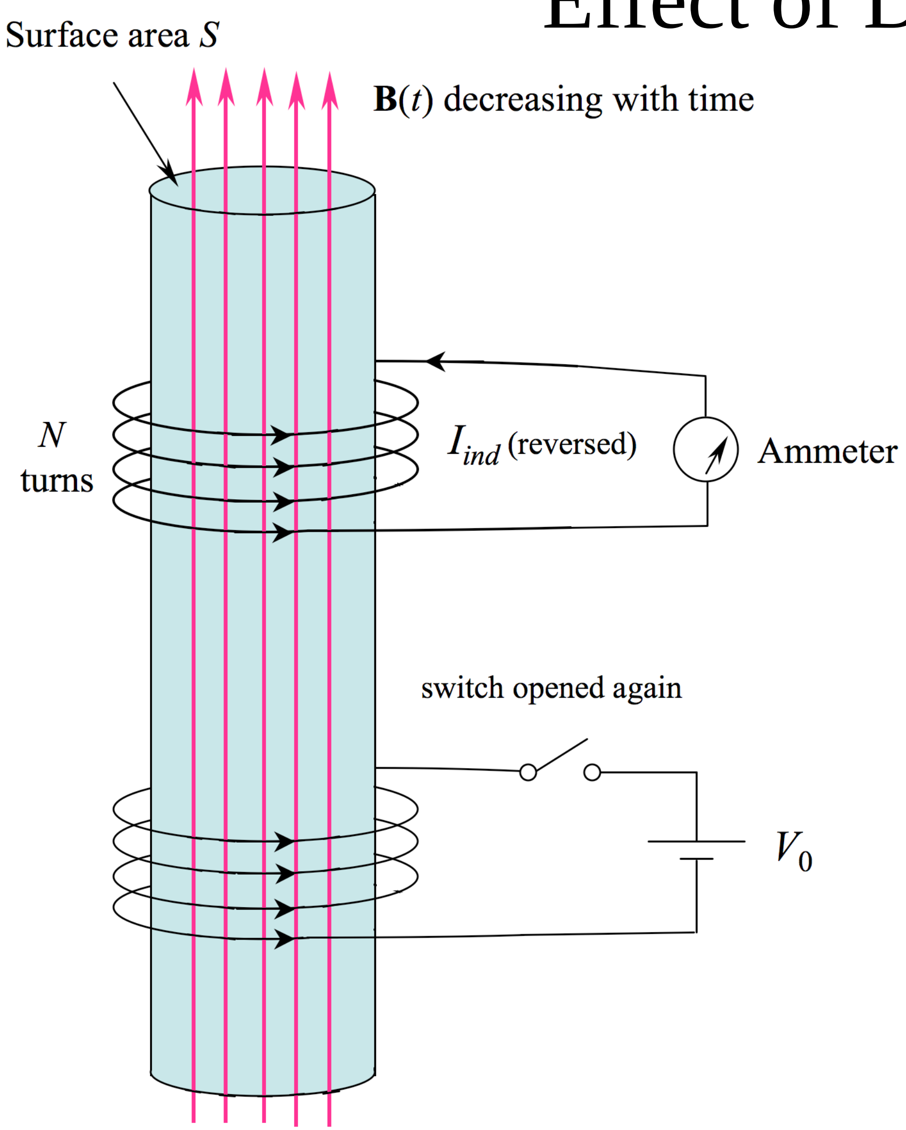
More specifically, the induced current is proportional to the time rate of change of the *magnetic flux* (the surface integral of **B** over S).

Steady State



Once steady state is reached, in which **B** is at a constant value, the induced current is found to be *zero*.

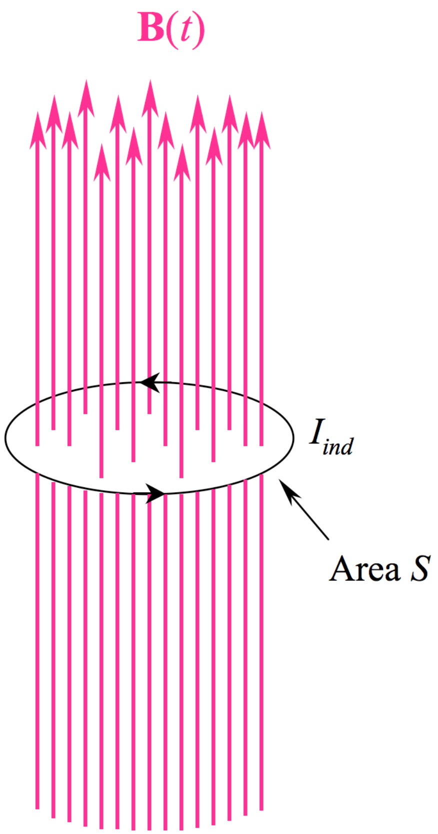
Effect of Decreasing **B**



Opening the switch results in current that initially decreases with time in the lower windings. Magnetic flux density **B** collapses, and the upper windings carry current, I_{ind} in the opposite direction from before. This current exists *as long as the magnetic flux decreases with time.*

Once **B** is reduced completely to zero, the induced current is zero as well.

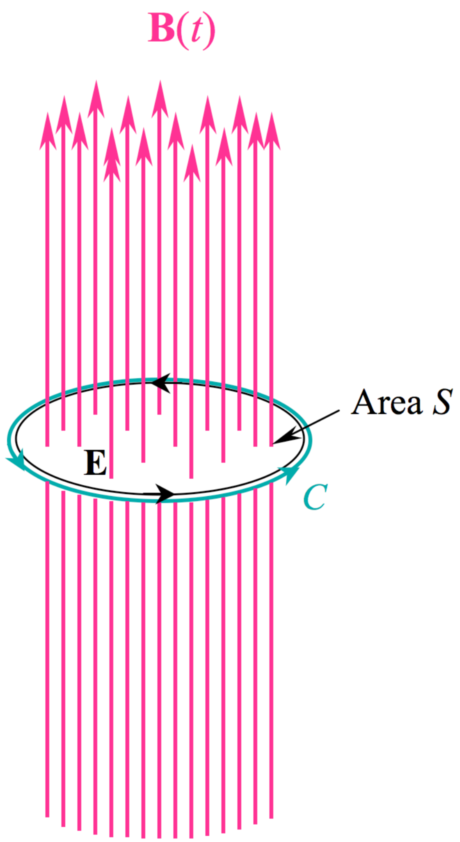
A Simpler Demonstration



Consider a single turn of wire, through which an externally-applied magnetic flux is present. The flux varies with time.

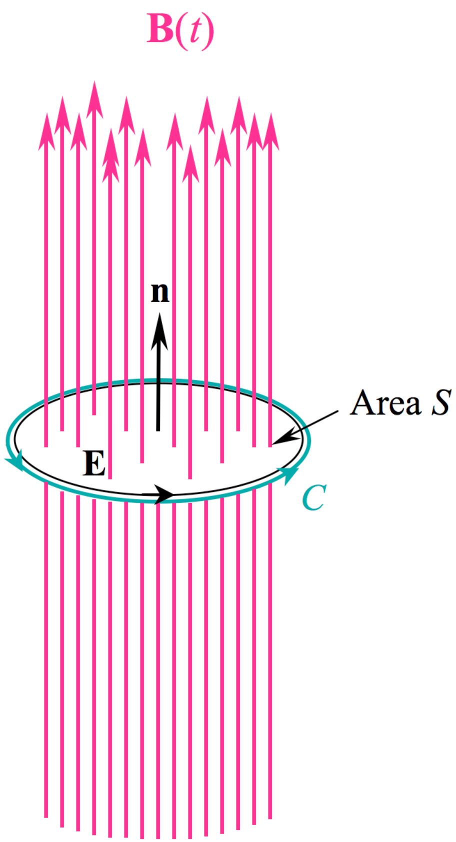
A current, I_{ind} , is generated in the wire loop as a result of the changing magnetic flux.

Replacing Current with Electric Field



Current in the wire is caused by electric field, **E**, that pushes the charge around the wire. In this illustration, the wire is removed, and the electric field remains. An integration contour, *C*, is also shown, that coincides with **E**.

Electromotive Force (emf)

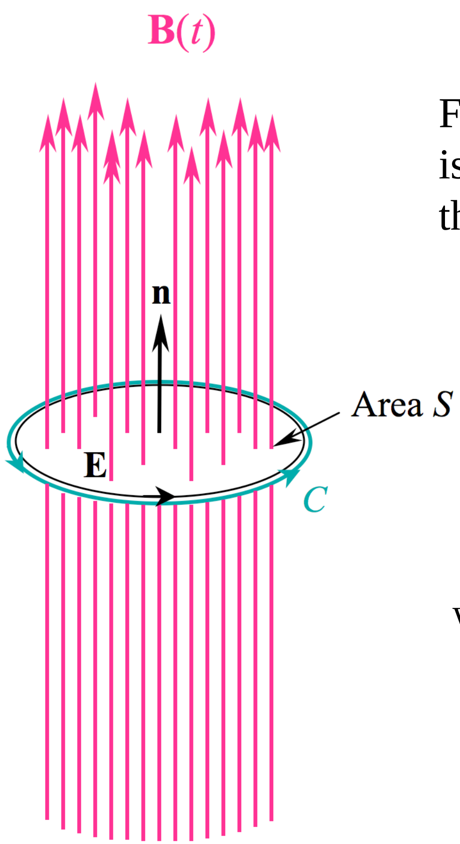


Next, a unit normal vector to the surface, **n**, is shown. The relative directions of **n**, and the contour C are defined by the right-hand convention: Right hand thumb in direction of **n**; fingers then curve in the direction of C.

The *electromotive force*, or *emf*, is defined as the closed path integral of **E** about C:

$$\text{emf} = \oint_C \mathbf{E} \cdot d\mathbf{L}$$

Faraday's Law of Induction



Faraday's Law states that the induced emf around a closed path is equal to the time rate of change of the magnetic flux through the area surrounded by the path:

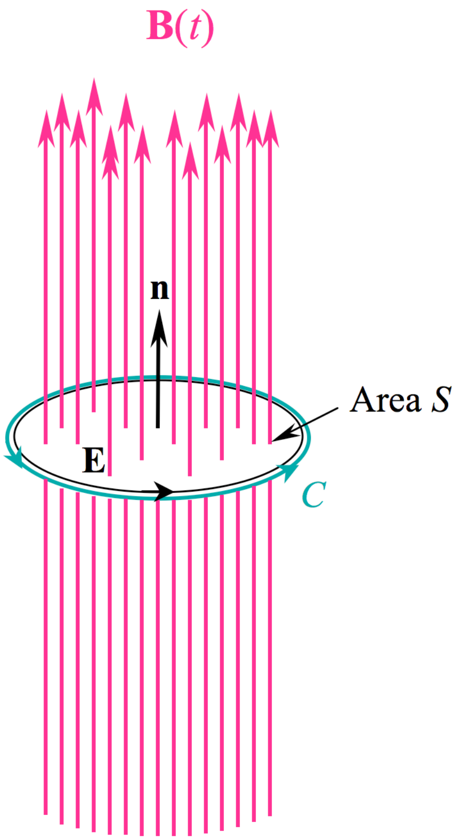
$$\oint_C \mathbf{E} \cdot d\mathbf{L} = -\frac{d\Phi}{dt}$$

where the flux is:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \mathbf{B} \cdot \mathbf{n}\, da$$

Note the use of the normal unit vector here, which determines the sign of the flux, and ultimately the emf.

Faraday’s Law in Detail



Using the definitions of emf and the magnetic flux in this situation:

$$\oint_C \mathbf{E} \cdot d\mathbf{L} = -\frac{d\Phi}{dt}$$

becomes:

$$\text{emf} = \oint_C \mathbf{E} \cdot d\mathbf{L} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n}\, da$$

Question:

When we strictly apply this rule to the illustration at the left,
do we conclude that **B** is increasing or decreasing with time?

Alternative Expression for Faraday's Law (Another Maxwell Equation)

First move the time derivative into the interior of the right-hand integral: ⟹

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Second, use Stokes' theorem to write the line integral of **E** as the surface integral of its curl: ⟹

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

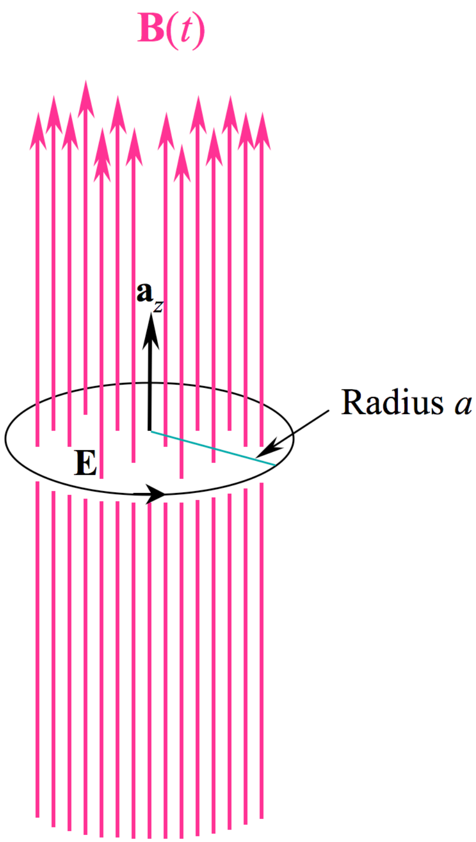
Third, simplify by noting that both integrals are taken over the same surface: ⟹

$$(\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Finally, obtain the point form of Faraday's Law: ⟹

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Example



Start with a uniform (but time-varying) field:

$$\mathbf{B} = B_0 e^{kt} \mathbf{a}_z$$

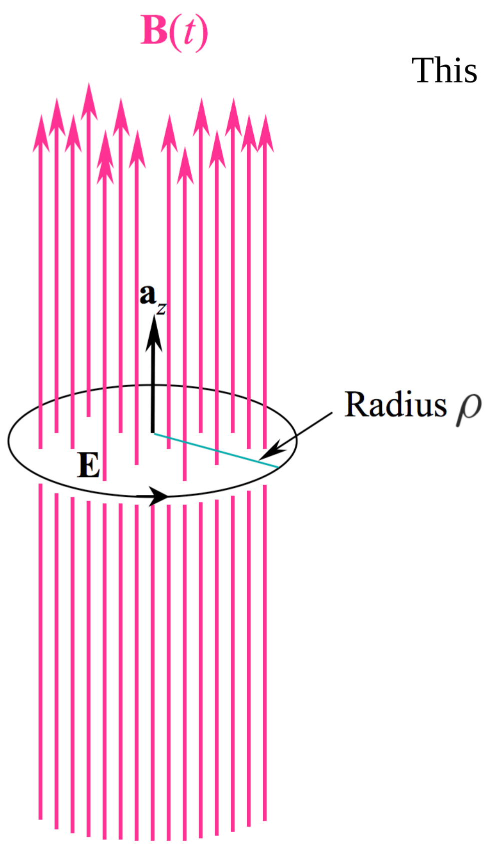
Then $$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = 2\pi a E_\phi$$

$$= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -k B_0 e^{kt} \pi a^2$$

As this calculation will work for any radius, we can replace the fixed radius with a variable radius and write:

$$\mathbf{E} = -\tfrac{1}{2} k B_0 e^{kt} \rho \mathbf{a}_\phi$$

Another Way…



This time use: $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ where $\mathbf{B} = B_0 e^{kt} \mathbf{a}_z$

Therefore: $(\nabla \times \mathbf{E})_z = -k B_0 e^{kt} = \underbrace{\dfrac{1}{\rho} \dfrac{\partial (\rho E_\phi)}{\partial \rho}}$

Surviving z-directed curl component with radial variation only

Multiply through by ρ and integrate both sides:

$$-\tfrac{1}{2} k B_0 e^{kt} \rho^2 = \rho E_\phi$$

Obtain, as before: $\mathbf{E} = -\tfrac{1}{2} k B_0 e^{kt} \rho \mathbf{a}_\phi$

An observation: The given **B** field does not satisfy all of Maxwell's equations! More about this later…

Induced emf from a Moving Closed Path

In the case shown here, the flux within a closed path is increasing with time as a result of the sliding bar, moving at constant velocity *v*.

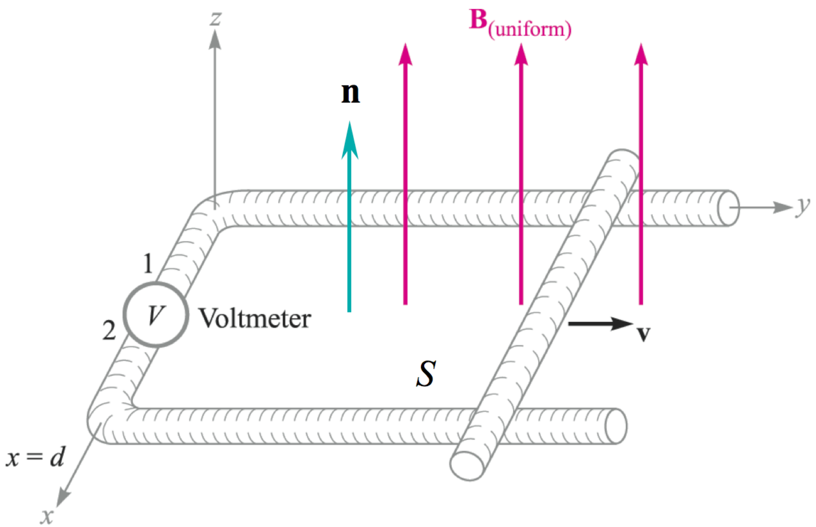
The conducting path is assumed perfectly-conducting, so that the induced emf appears entirely across the voltmeter.

The magnetic flux enclosed by the conducting path is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \mathbf{B} \cdot \mathbf{n}\, da$$

$$= Byd$$

where the *y* dimension varies with time.



Therefore: $$\text{emf} = -\frac{d\Phi}{dt} = -B\frac{dy}{dt}d = -Bvd$$

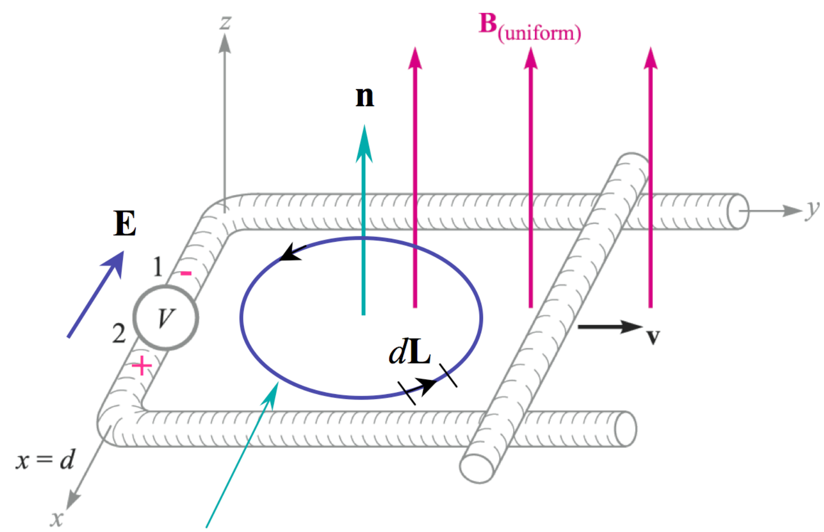
Finding the Direction of **E**

..so we have:

$$\text{emf} = -\frac{d\Phi}{dt} = \oint \mathbf{E} \cdot d\mathbf{L} = -Bvd$$

The **E** integration path direction is determined by the right hand convention (thumb of right hand in direction of **n**,fingers in direction of closed path). This is the direction of *d***L** as shown in the figure.

From the equation, we see that $\mathbf{E} \cdot d\mathbf{L}$ is negative, meaning that **E** will point from terminals 2 to 1 as shown. As we have a perfectly-conducing path, **E** will exist only at the voltmeter.



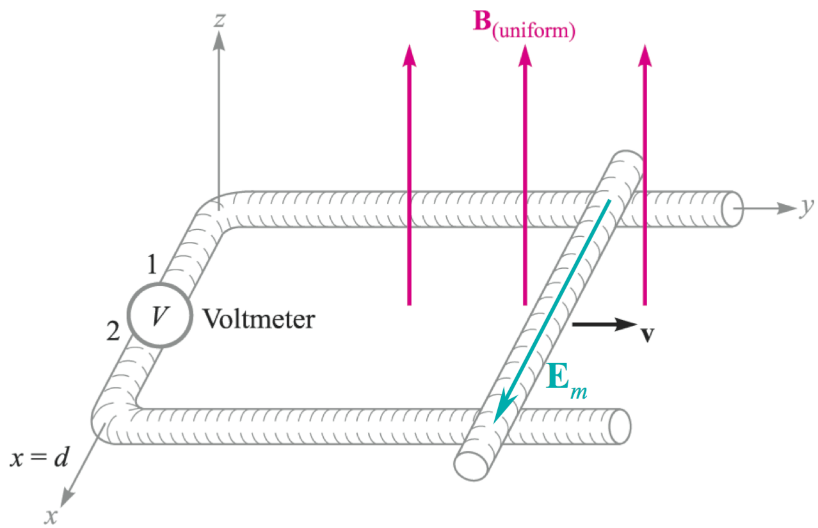
Direction of integration path of **E** through conductor (by right hand convention)

Motional EMF

The sliding bar contains free charges (electrons) that experience Lorentz forces as they move through the **B** field:

Recall: $$\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$$

and so.. $$\frac{\mathbf{F}}{Q} = \mathbf{v} \times \mathbf{B}$$



This result is the *motional field intensity:*

$$\mathbf{E}_m = \mathbf{v} \times \mathbf{B}$$

..and so now the *motional emf* is the closed path integral of $\mathbf{E}_m$ over the same path as before.

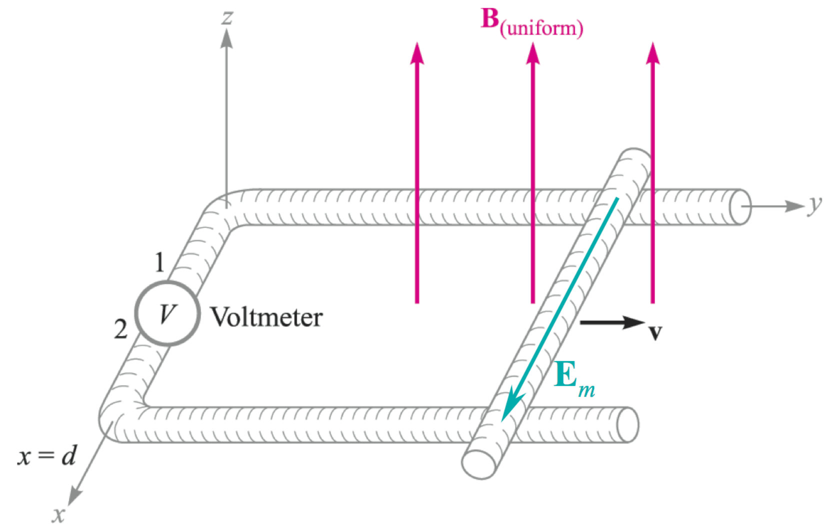
$$\text{emf} = \oint \mathbf{E}_m \cdot d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}$$

Note that the motional field intensity exists only in the part of the circuit that is in motion.

Reprise of the Original Problem

We now have the motional emf:

$$\text{emf} = \oint \mathbf{E}_m \cdot d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}$$



..which in the current problem becomes:

$$\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} = \int_d^0 vB\,dx = -Bvd$$

or the same result as before!

as always, this path integral is taken according to the right hand convention, leading to the ordering of limits shown here

Two Contributions to emf

For a moving conducting path in an otherwise uniform field, the emf is composed entirely of motional emf:

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = \oint \mathbf{E}_m \cdot d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}$$

When the **B** field is time-varying as well, then both effects need to be taken into account. The total emf is:

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = \underbrace{-\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}}_{\text{Transformer emf}} + \underbrace{\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}}_{\text{Motional emf}}$$

Transformer emf Motional emf

Together, the two effects are really just... $-\dfrac{d\Phi}{dt}$

An Apparent Contradiction

We now have two the curl equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \nabla \times \mathbf{H} = \mathbf{J}$$

Faraday's Law   Ampere's Circuital Law

Taking the divergence of the second equation gives us a result that is identically zero:

$$\nabla \cdot \nabla \times \mathbf{H} \equiv 0 = \nabla \cdot \mathbf{J}$$

But a previous exercise gave us this result: the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

So what went wrong?

The Missing Term -- Displacement Current Density

Suppose we add a term to the right-hand side of Ampere's Law, whose identity we need to determine:

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{G}$$

Added term to be identified

Now take the divergence:

$$\nabla \cdot \nabla \times \mathbf{H} \equiv 0 = \underbrace{\nabla \cdot \mathbf{J}}_{-\frac{\partial \rho_\nu}{\partial t}} + \underbrace{\nabla \cdot \mathbf{G}}_{\frac{\partial \rho_\nu}{\partial t}}$$

From the equation of continuity → $-\dfrac{\partial \rho_\nu}{\partial t}$

This must be true: → $\dfrac{\partial \rho_\nu}{\partial t}$

So now: $$\nabla \cdot \mathbf{G} = \frac{\partial \rho_\nu}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

..from which: $$\mathbf{G} = \frac{\partial \mathbf{D}}{\partial t}$$

Now, Ampere's Law is modified: $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

This term is the *displacement current density*

Symmetry in the Curl Equations

Our new result states that the total current is composed of conduction current and displacement current:

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

where the displacement current density is:

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$$

Conduction current density is: $\mathbf{J} = \sigma \mathbf{E}$ as before.

In the absence of conduction current, and when fields are time-varying, we have:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad (\text{if } \mathbf{J} = 0)$$

Note the symmetry when comparing to Faraday's Law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Displacement Current in Ampere's Law

Displacement current is found from displacement current density by taking the appropriate surface integral:

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{S} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

Such an integral of Ampere's Circuital Law curl equation gives:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

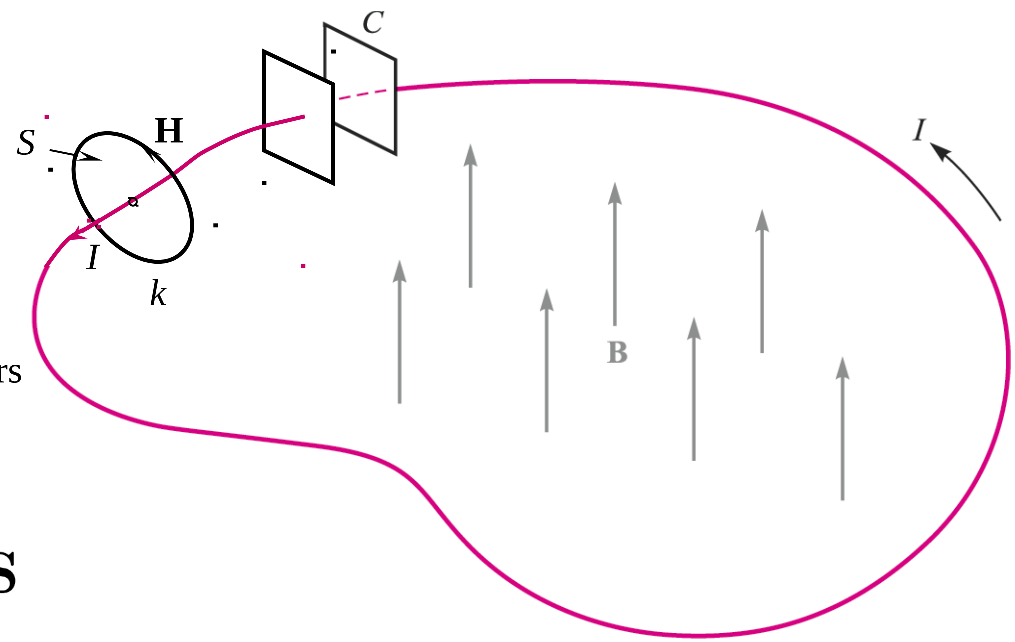
Applying Stokes' Theorem to the left side gives the more familiar form of Ampere's Law, now with displacement current added:

$$\oint \mathbf{H} \cdot d\mathbf{L} = I + I_d = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

Demonstration of Displacement Current

In this situation, a time-varying magnetic field links a wire loop that is connected to a parallel-plate capacitor.

The magnetic field is presumed time-varying, and thus it generates emf, which in turn provides the current.



We assume for the moment that conduction current *I* exists in the wire. Ampere's Law is applied to the circular path shown, whose radius is very small compared to the local wire length. The wire therefore appears very long and straight. We obtain:

$$\oint_k \mathbf{H} \cdot d\mathbf{L} = \int\int_S \mathbf{J} \cdot d\mathbf{S}$$

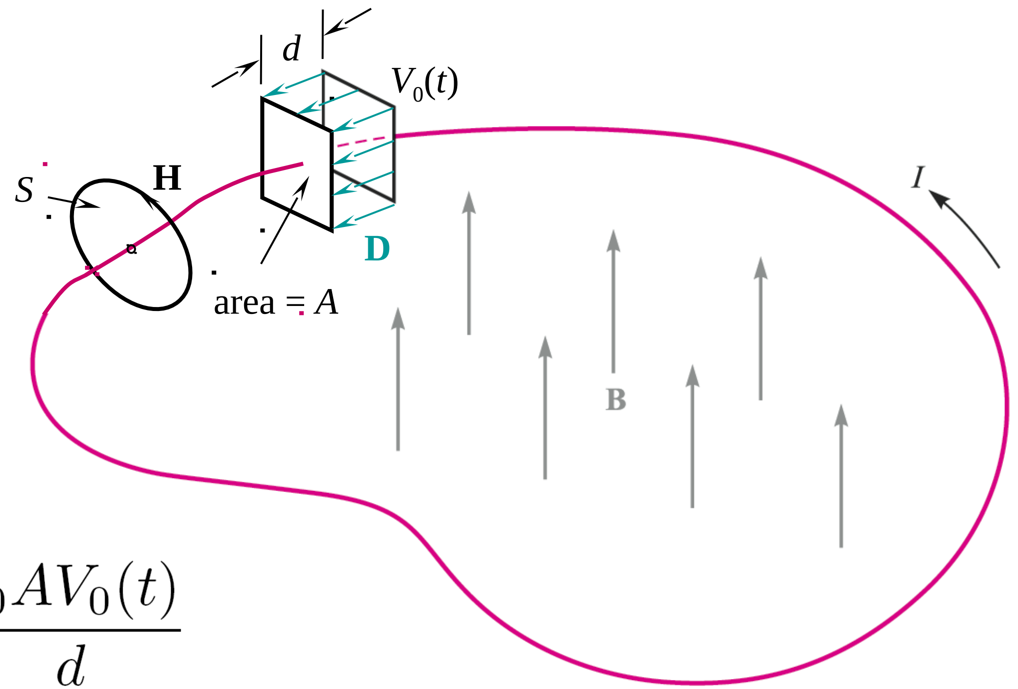
$$\Rightarrow \mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$ as usual

The integration surface is the flat plane whose perimeter is the circular path, *k*. We assign the *z* direction (locally) to be along the current where it passes through the plane.

Demonstration of Displacement Current

While current is flowing, the electric field in the capacitor is changing with time, in step with the plate charge level, deposited by the current.

The capacitor voltage is $V_0(t)$. Taking the capacitor axis as z, assuming free space permittivity, and with cap dimensions as indictated, the time-varying electric flux density within it is:

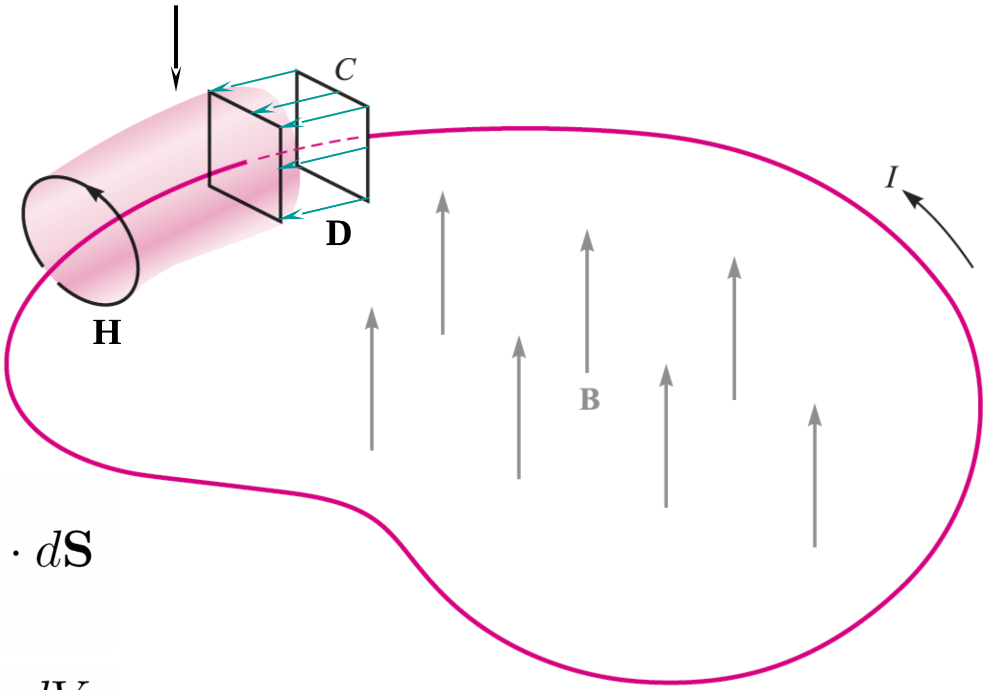


$$\Phi_e = \int\int \mathbf{D} \cdot d\mathbf{S} = \frac{\epsilon_0 A V_0(t)}{d}$$

Demonstration of Displacement Current

Now suppose the surface S, surrounded by the original contour, k, is stretched so that it intercepts the electric field between capacitor plates:

Stretched surface now intercepts electric field lines, and *no conduction current*



Ampere's Law is again applied, but this time we need to use the displacement current in the capacitor:

$$\oint_k \mathbf{H} \cdot d\mathbf{L} = I_d = \int\int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

$$= \frac{d\Phi_e}{dt} = \left(\frac{\epsilon_0 A}{d}\right) \frac{dV_0}{dt} = C \frac{dV_0}{dt}$$

The result must be independent of the surface that is chosen, which leads to the conclusion that the conduction and displacement currents must be equal:

$$I = C \frac{dV_0}{dt}$$

a familiar result!

Maxwell's Equations in Point Form

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Ampere's Circuital Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's Law of Induction

$$\nabla \cdot \mathbf{D} = \rho_v$$

Gauss' Law for the electric field

$$\nabla \cdot \mathbf{B} = 0$$

Gauss's Law for the magnetic field

Ampere's Circuital Law in Integral Form

Take the surface integral of the point form equation:

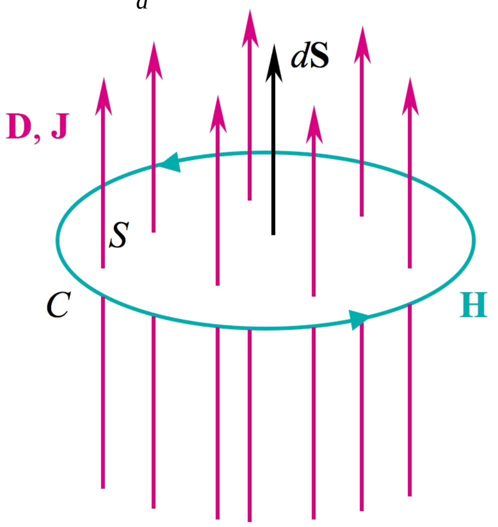
$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \underbrace{\int_S \mathbf{J} \cdot d\mathbf{S}}_{I} + \underbrace{\int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}}_{I_d}$$

Apply Stokes's Theorem to the left hand side:

$$\underbrace{\oint_C \mathbf{H} \cdot d\mathbf{L}}_{\text{mmf}} = I + I_d$$

where

$$I_d = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} = \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S} = \frac{d\Phi_e}{dt}$$



Remember to use the right hand convention on relating path direction to the normal vector.

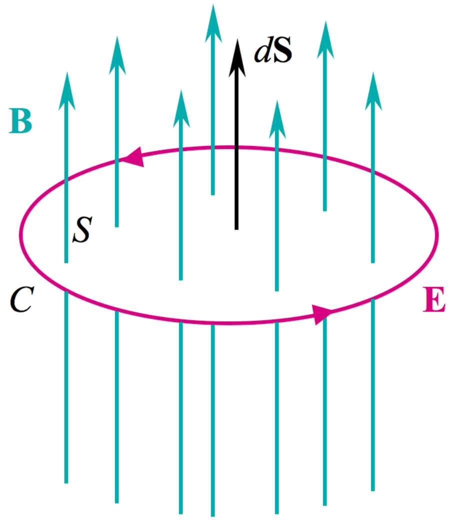
Faraday's Law of Induction in Integral Form

Take the surface integral of the point form equation:

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \int_S -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Apply Stokes's Theorem to the left hand side:

$$\underbrace{\oint_C \mathbf{E} \cdot d\mathbf{L}}_{\text{emf}} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} = -\frac{d\Phi_m}{dt}$$



Remember to use the right hand convention on relating path direction to the normal vector.

Gauss's Law for the Electric Field in Integral Form

Begin with: $\nabla \cdot \mathbf{D} = \rho_v$ and integrate both sides over a volume, *v*:

$$\int_v \nabla \cdot \mathbf{D}\, dv = \int_v \rho_v\, dv = Q_{encl}$$

Apply the divergence theorem to the left hand side:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{encl}$$

Gauss's Law for the Magnetic Field in Integral Form

Begin with: $\nabla \cdot \mathbf{B} = 0$ and integrate both sides over a volume, *v*:

$$\int_v \nabla \cdot \mathbf{B} \, dv = 0$$

Apply the divergence theorem to the left hand side:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Scalar and Vector Potentials, Revisited

In a previous unit, we defined the scalar electric potential, and vector magnetic potential, under conditions in which there is no time variation:

Scalar electric potential (satisfies Poisson's equation):

$$V = \int_{\text{vol}} \frac{\rho_v \, dv}{4\pi\epsilon R} \quad \text{(static)} \qquad\qquad \nabla^2 V = -\frac{\rho_v}{\epsilon}$$

Vector magnetic potential:

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu \mathbf{J} \, dv}{4\pi R} \quad \text{(dc)} \qquad\qquad \nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

Knowing *V*, we find **E** through: $\mathbf{E} = -\nabla V \quad \text{(static)}$

Knowing **A**, we find **B** through: $\mathbf{B} = \nabla \times \mathbf{A} \quad \text{(dc)}$

Inconsistancies with Time-Varying Fields

Take the curl of both sides of $\mathbf{E} = -\nabla V$

$$\nabla \times \mathbf{E} = \underbrace{\nabla \times (-\nabla V)}_{\text{This is identically zero!}} = \underbrace{-\frac{\partial \mathbf{B}}{\partial t}}_{\text{This is not}}$$

So instead, postulate a correction to the original equation, where the field **N** is to be found:

$$\mathbf{E} = -\nabla V + \mathbf{N}$$

Now take the curl:

$$\nabla \times \mathbf{E} = 0 + \nabla \times \mathbf{N} = -\frac{\partial \mathbf{B}}{\partial t}$$

Inconsistancies with Time-Varying Fields

We now have: $$\nabla \times \mathbf{E} = 0 + \nabla \times \mathbf{N} = -\frac{\partial \mathbf{B}}{\partial t}$$

where $$\mathbf{B} = \nabla \times \mathbf{A}$$

Therefore: $$\nabla \times \mathbf{N} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

or: $$\nabla \times \mathbf{N} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t} \quad \Rightarrow \quad \mathbf{N} = -\frac{\partial \mathbf{A}}{\partial t}$$

So the modified expression for
E under time-varying conditions is:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

Further Confirmation: Satisfying the Other Maxwell Equations

We must now verify that $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ satisfies the other Maxwell Equations

To do this, begin by substituting the equation into the other Maxwell curl equation:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Using: $\begin{cases} \mathbf{B} = \nabla \times \mathbf{A} = \mu \mathbf{H} \\ \mathbf{D} = \epsilon \mathbf{E} \end{cases}$

Obtain:

$$\frac{1}{\mu} \nabla \times \nabla \times \mathbf{A} = \mathbf{J} + \epsilon \left(-\nabla \frac{\partial V}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right)$$

Satisfying Ampere's Law (continued)

Now have:

$$\frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} = \mathbf{J} + \epsilon\left(-\nabla\frac{\partial V}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2}\right)$$

Use the vector identity: $\nabla \times \nabla \times \mathbf{A} \ = \ \nabla(\nabla \cdot \mathbf{A}) \ - \ \nabla^2 \mathbf{A}$

to obtain:

$$\nabla(\nabla \cdot \mathbf{A}) \ - \ \nabla^2 \mathbf{A} \ = \ \mu \mathbf{J} \ - \ \mu\epsilon\left(\nabla\frac{\partial V}{\partial t} \ + \ \frac{\partial^2 \mathbf{A}}{\partial t^2}\right)$$

Satisfying Gauss' Law

Next: $$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

must satisfy $\nabla \cdot \mathbf{D} = \rho_\nu$ where $\mathbf{D} = \epsilon \mathbf{E}$

Make the substitution to find:

$$\epsilon\left(-\nabla \cdot \nabla V - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A}\right) = \rho_\nu$$

or finally: $$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho_\nu}{\epsilon}$$

Collecting Results

From Ampere's Law, we now have:

$$\nabla(\nabla \cdot \mathbf{A}) \;-\; \nabla^2 \mathbf{A} \;=\; \mu \mathbf{J} \;-\; \mu\epsilon \left(\nabla \frac{\partial V}{\partial t} \;+\; \frac{\partial^2 \mathbf{A}}{\partial t^2} \right)$$

.. and from Gauss' Law:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho_\nu}{\epsilon}$$

Assigning the Divergence: Lorentz Gauge

A vector field is fully characterized if we have specified its divergence, its curl, its value at a specific point, which could lie at infinity. The vector potential, **A**, already has its curl specified (**B**), so we are now free to specify the divergence in such a way as to make the equations sensible (and solvable) for real situations.

In view of these considerations, we specify the divergence of **A** as follows:

$$\nabla \cdot \mathbf{A} \;=\; -\mu\epsilon\frac{\partial V}{\partial t}$$

Such assignments are known as *gauge conditions*. The one just identified is known as the *Lorentz Gauge*.

Simplifying the Equations

With the Lorentz Gauge, $\nabla \cdot \mathbf{A} = -\mu\epsilon \dfrac{\partial V}{\partial t}$

the two equations are simplified in the following ways:

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu\epsilon \left(\nabla \frac{\partial V}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right)$$

becomes:

$$\boxed{\nabla^2 \mathbf{A} = -\mu \mathbf{J} + \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}}$$

and the previous:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho_v}{\epsilon}$$

now becomes:

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon} + \mu\epsilon \frac{\partial^2 V}{\partial t^2}}$$

These are both *wave equations*, yielding propagating functions as their solutions. This topic will be discussed at length in subsequent chapters.

The Propagation Effect: Delayed Response

The ulimate effect of time-varying fields is that their effects (with regard to potentials, forces, or their very existence) will be felt after a *time delay*. This delay is proportional to the distance between the source of the field, and the point of observation, and represents the time required for a "disturbance" to propagate between those two points. So the previous integral expressions that we had for scalar and vector potential are modifed to accommodate this delay in a fairly simple manner:

Static Potentials

Retarded Potentials

$$V = \int_{\text{vol}} \frac{\rho_\nu d\nu}{4\pi\epsilon R} \quad \longrightarrow \quad V = \int_{vol} \frac{\rho_v\,(t - R/v_p)}{4\pi\epsilon R}\, dv$$

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu \mathbf{J}\, d\nu}{4\pi R} \quad \longrightarrow \quad \mathbf{A} = \int_{vol} \frac{\mu\,\mathbf{J}\,(t - R/v_p)}{4\pi R}\, dv$$

v_p is the "propagation velocity" (not to be confused with volume). *V* and **A** are now propagating functions.

The argument $(t - R/v_p)$ indicates that at farther distances, *R*, the delay time *t* at which the potential will be felt is longer.