Engineering Electromagnetics

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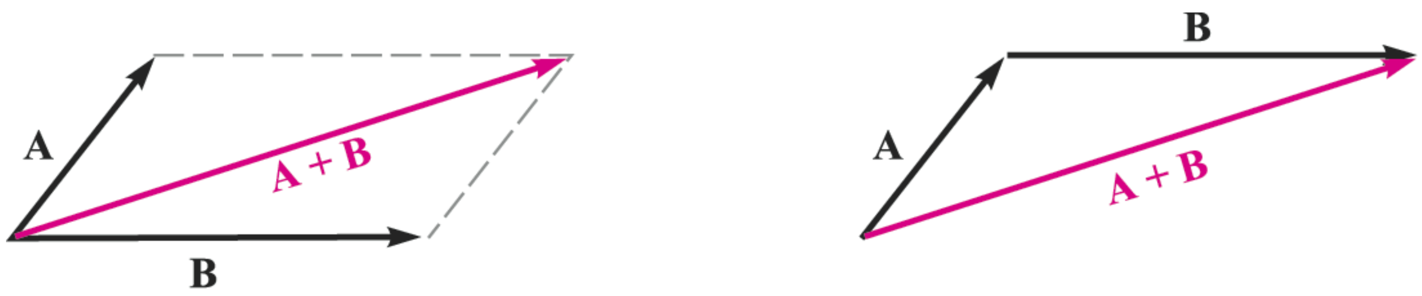
Chapter 1:

Vector Analysis

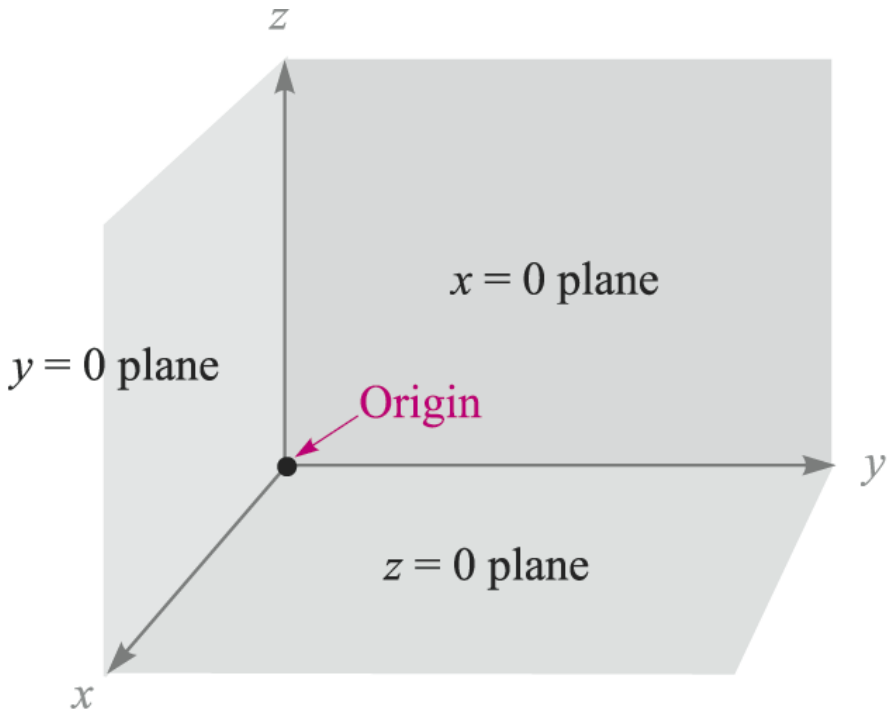
Vector Addition



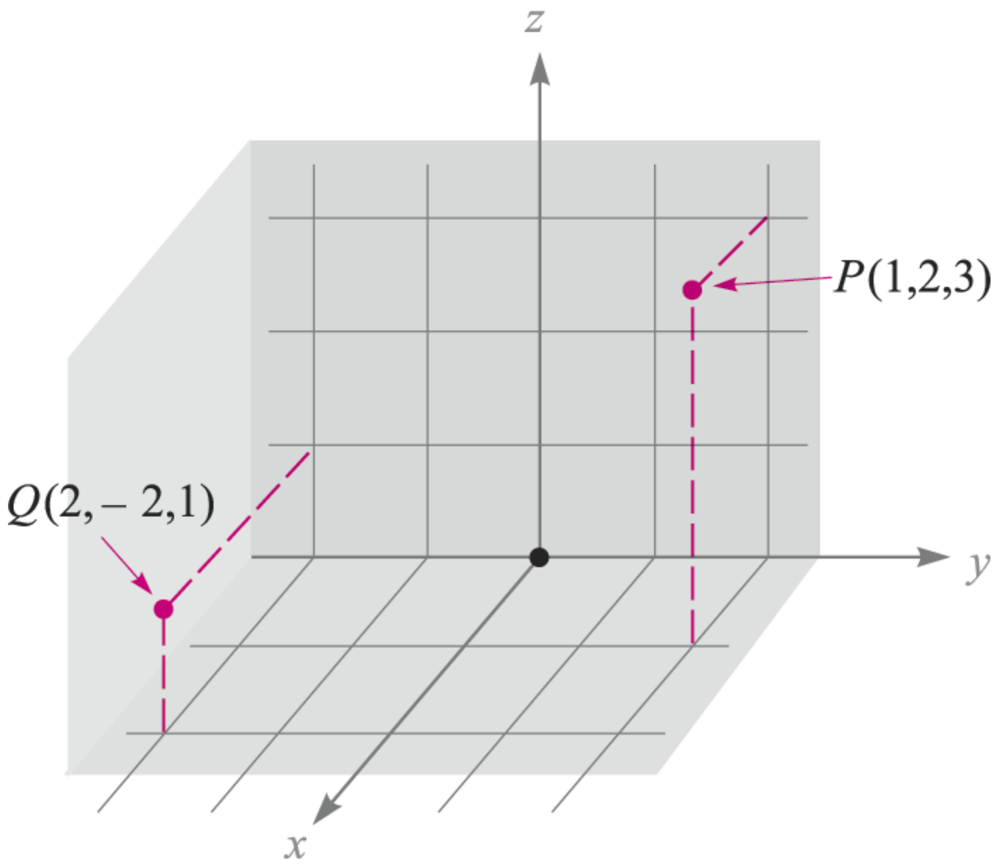
Associative Law: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

Distributive Law: $(r + s)(\mathbf{A} + \mathbf{B}) = r(\mathbf{A} + \mathbf{B}) + s(\mathbf{A} + \mathbf{B})$

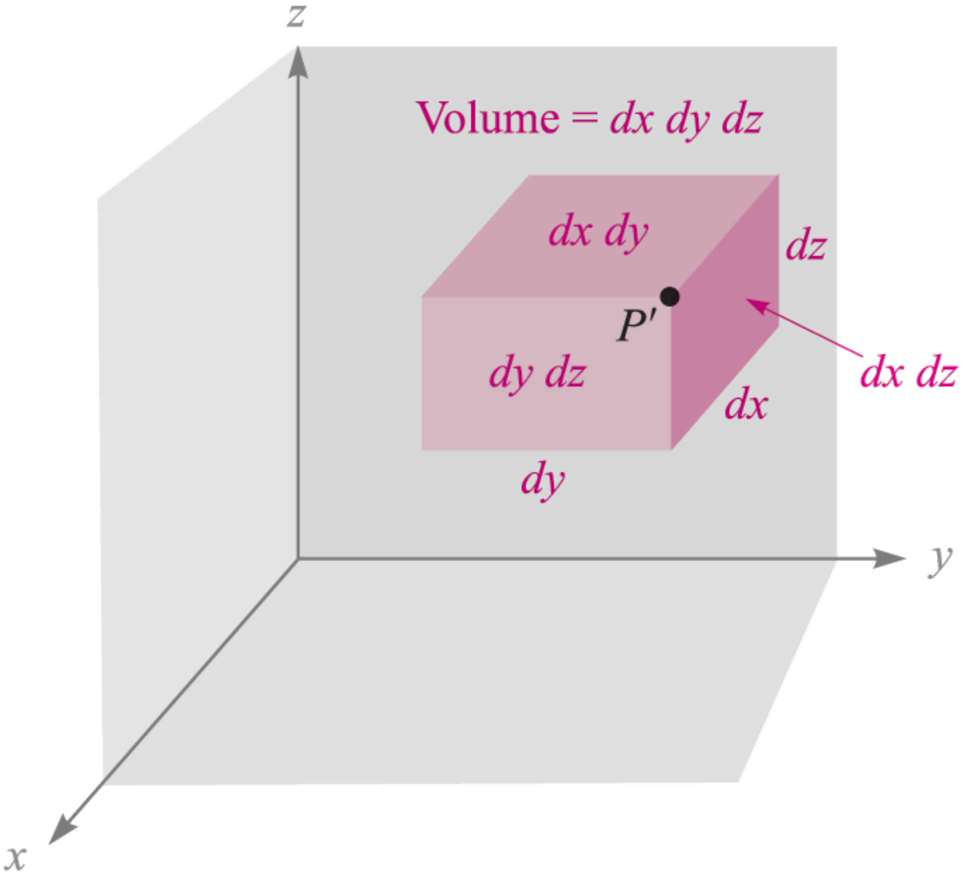
Rectangular Coordinate System



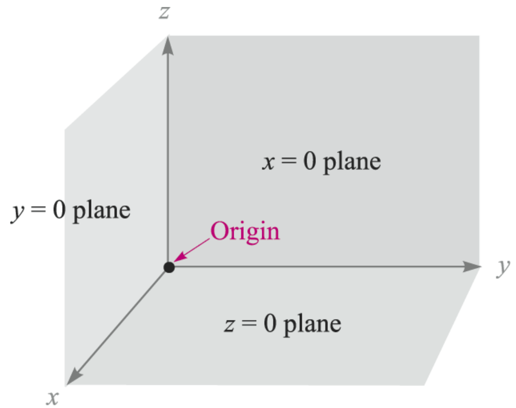
Point Locations in Rectangular Coordinates



Differential Volume Element

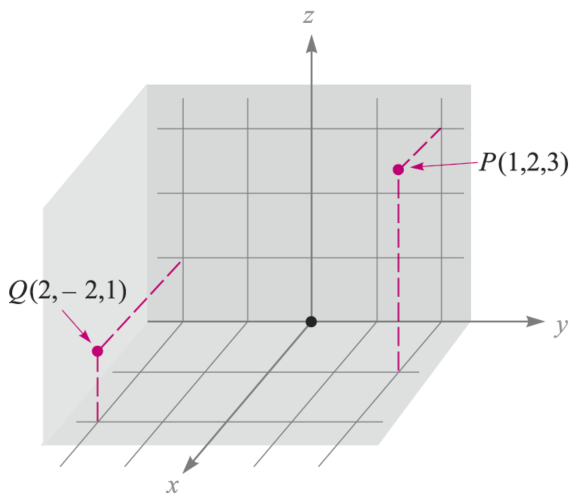


Summary



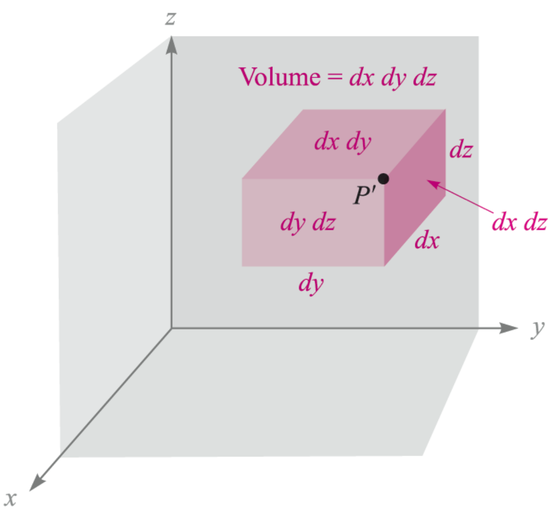
z
x = 0 plane
y = 0 plane
Origin
y
z = 0 plane
x

(a)



z
P(1,2,3)
Q(2, – 2,1)
y
x

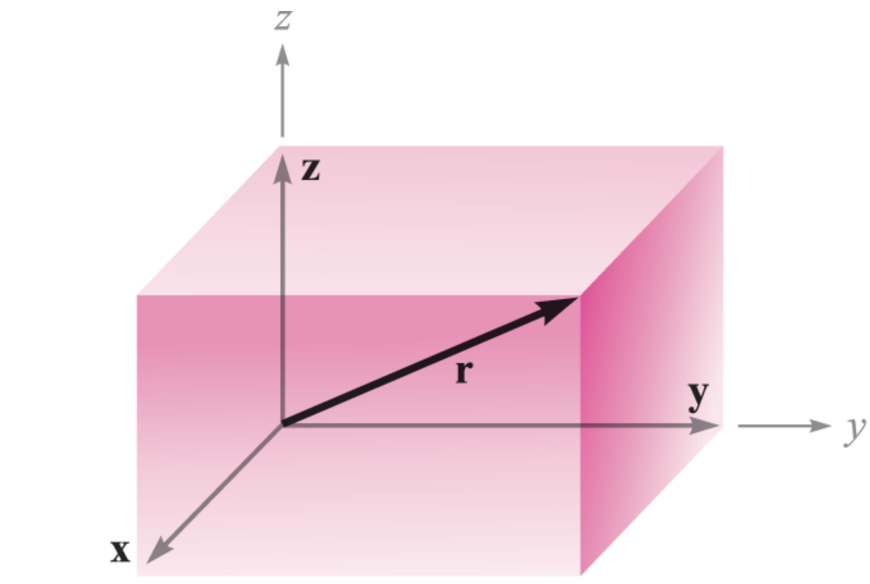
(b)



z
Volume = dx dy dz
dx dy
dz
P′
dy dz
dx dz
dx
dy
y
x

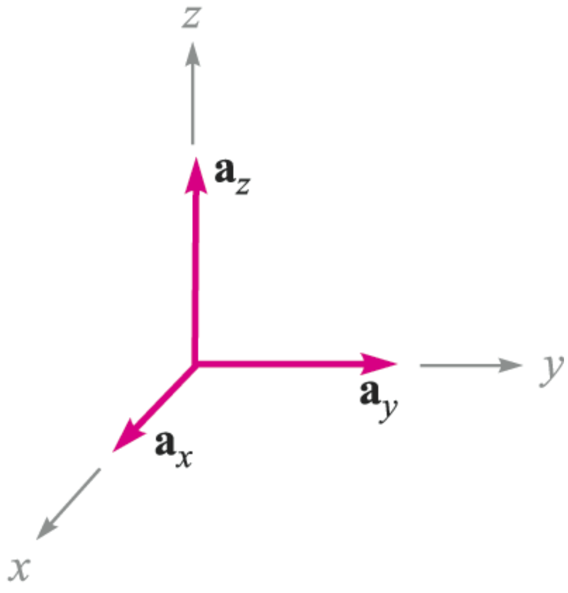
(c)

Orthogonal Vector Components

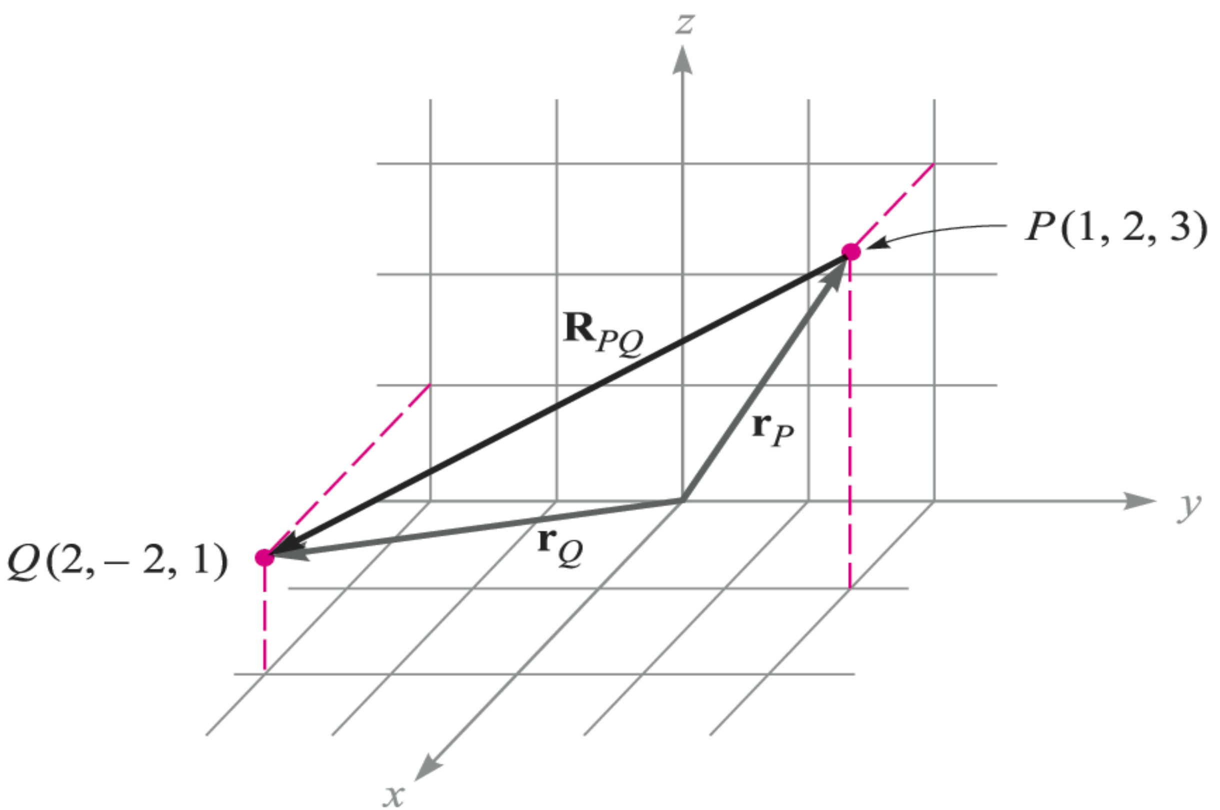


$\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$

Orthogonal Unit Vectors

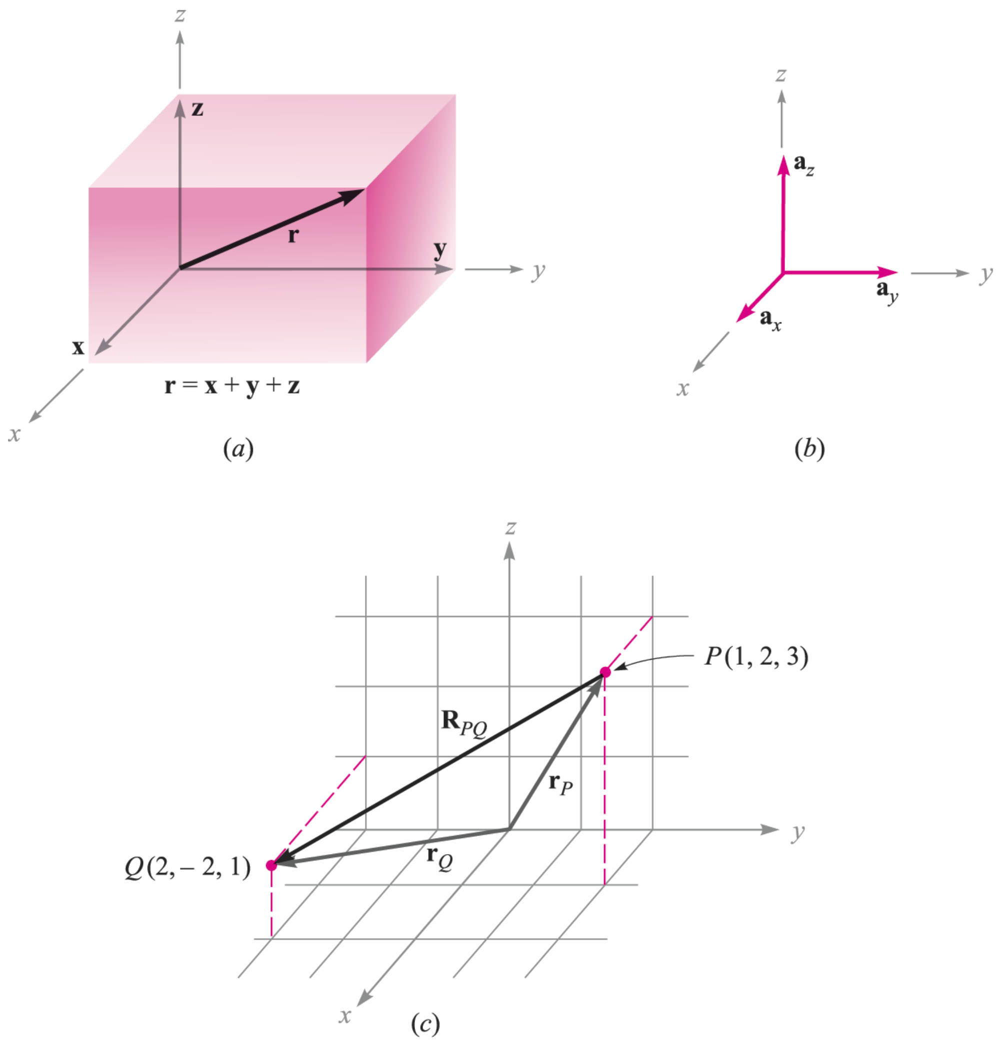


Vector Representation in Terms of Orthogonal Rectangular Components



$$\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z$$
$$= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$

Summary



z
z
r
y
y
x
x
$\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$
(a)
z
$\mathbf{a}_z$
$\mathbf{a}_y$
y
$\mathbf{a}_x$
x
(b)
z
$P(1, 2, 3)$
$\mathbf{R}_{PQ}$
$\mathbf{r}_P$
y
$Q(2, -2, 1)$
$\mathbf{r}_Q$
x
(c)

Vector Expressions in Rectangular Coordinates

General Vector, **B**:

$$\mathbf{B} = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z$$

Magnitude of **B**:

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Unit Vector in the Direction of **B:**

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|}$$

Example

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \tfrac{2}{3}\mathbf{a}_x - \tfrac{2}{3}\mathbf{a}_y - \tfrac{1}{3}\mathbf{a}_z = \underline{0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z}$$

Vector Field

We are accustomed to thinking of a specific vector:

$$\mathbf{v} = v_x \mathbf{a}_x + v_y \mathbf{a}_y + v_z \mathbf{a}_z$$

A vector field is a *function* defined in space that has magnitude and direction at all points:

$$\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})\mathbf{a}_x + v_y(\mathbf{r})\mathbf{a}_y + v_z(\mathbf{r})\mathbf{a}_z$$

where $\mathbf{r} = (x,y,z)$

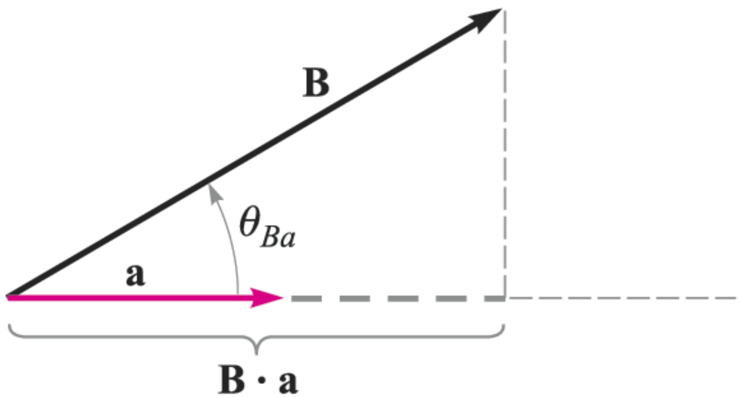
The Dot Product

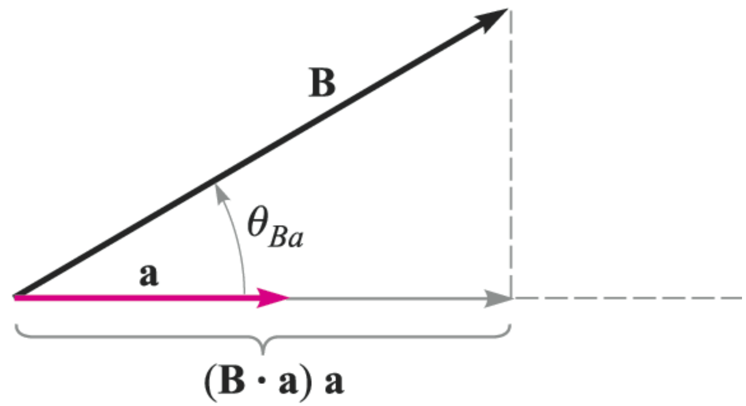
Given two vectors **A** and **B**, the *dot product*, or *scalar product*, is defined as the product of the magnitude of **A**, the magnitude of **B**, and the cosine of the smaller angle between them,

$$\boxed{\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta_{AB}}$$

Commutative Law: $\boxed{\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}}$

Vector Projections Using the Dot Product





B • a gives the component of **B** in the horizontal direction

(**B • a**)**a** gives the *vector* component of **B** in the horizontal direction

Operational Use of the Dot Product

Given
$$\begin{cases} \mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z \\ \mathbf{B} = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z \end{cases}$$

Find
$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z}$$

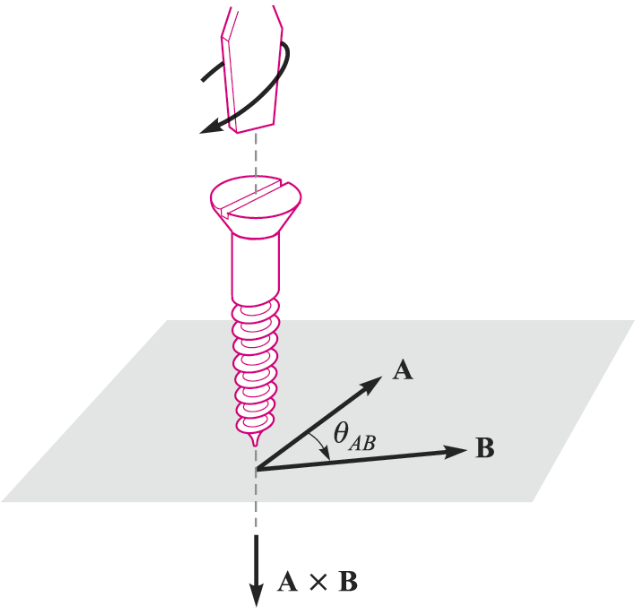
where we have used:
$$\begin{cases} \mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_x \cdot \mathbf{a}_z = 0 \\ \mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1 \end{cases}$$

Note also:
$$\boxed{\mathbf{A} \cdot \mathbf{A} = A^2 = |\mathbf{A}|^2}$$

Cross Product

The cross product $\mathbf{A} \times \mathbf{B}$ is a vector; the magnitude of $\mathbf{A} \times \mathbf{B}$ is equal to the product of the magnitudes of $\mathbf{A}$, $\mathbf{B}$, and the sine of the smaller angle between $\mathbf{A}$ and $\mathbf{B}$; the direction of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$ and is along that one of the two possible perpendiculars which is in the direction of advance of a right-handed screw as $\mathbf{A}$ is turned into $\mathbf{B}$.

$$\boxed{\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}|\,|\mathbf{B}| \sin\theta_{AB}}$$



Operational Definition of the Cross Product in Rectangular Coordinates

Begin with:

$$\begin{aligned}\mathbf{A} \times \mathbf{B} = {} & A_x B_x \mathbf{a}_x \times \mathbf{a}_x + A_x B_y \mathbf{a}_x \times \mathbf{a}_y + A_x B_z \mathbf{a}_x \times \mathbf{a}_z \\ & + A_y B_x \mathbf{a}_y \times \mathbf{a}_x + A_y B_y \mathbf{a}_y \times \mathbf{a}_y + A_y B_z \mathbf{a}_y \times \mathbf{a}_z \\ & + A_z B_x \mathbf{a}_z \times \mathbf{a}_x + A_z B_y \mathbf{a}_z \times \mathbf{a}_y + A_z B_z \mathbf{a}_z \times \mathbf{a}_z\end{aligned}$$

$$\boxed{\text{where} \begin{cases} \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \\ \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x \\ \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y \end{cases}}$$

Therefore:

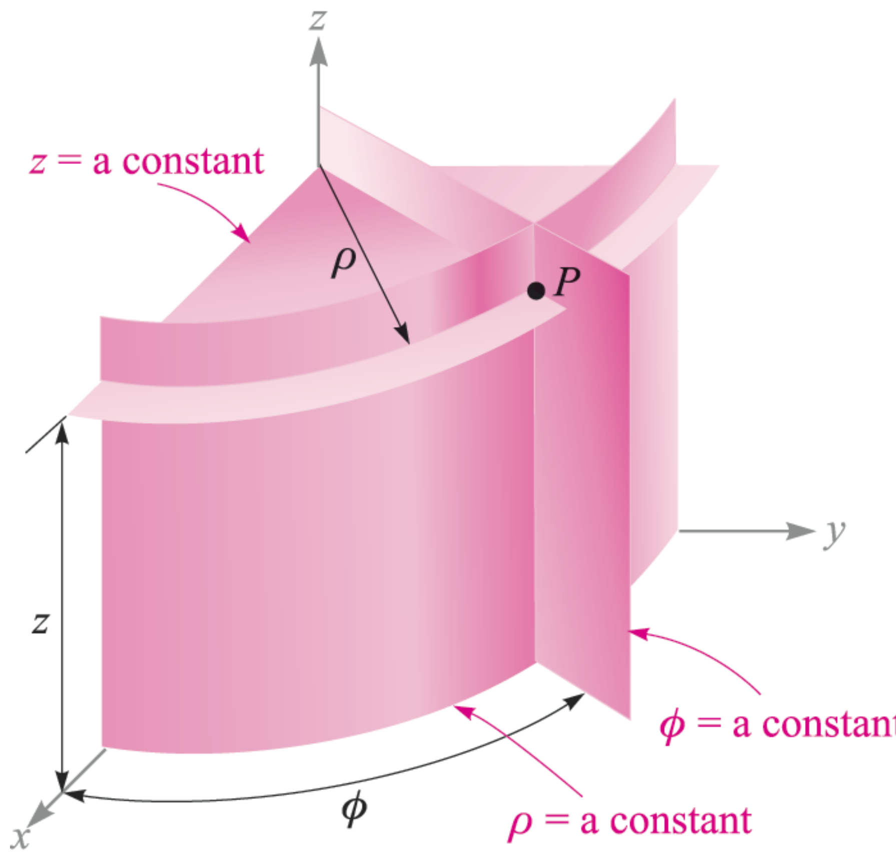
$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{a}_x + (A_z B_x - A_x B_z)\mathbf{a}_y + (A_x B_y - A_y B_x)\mathbf{a}_z$$

Or…

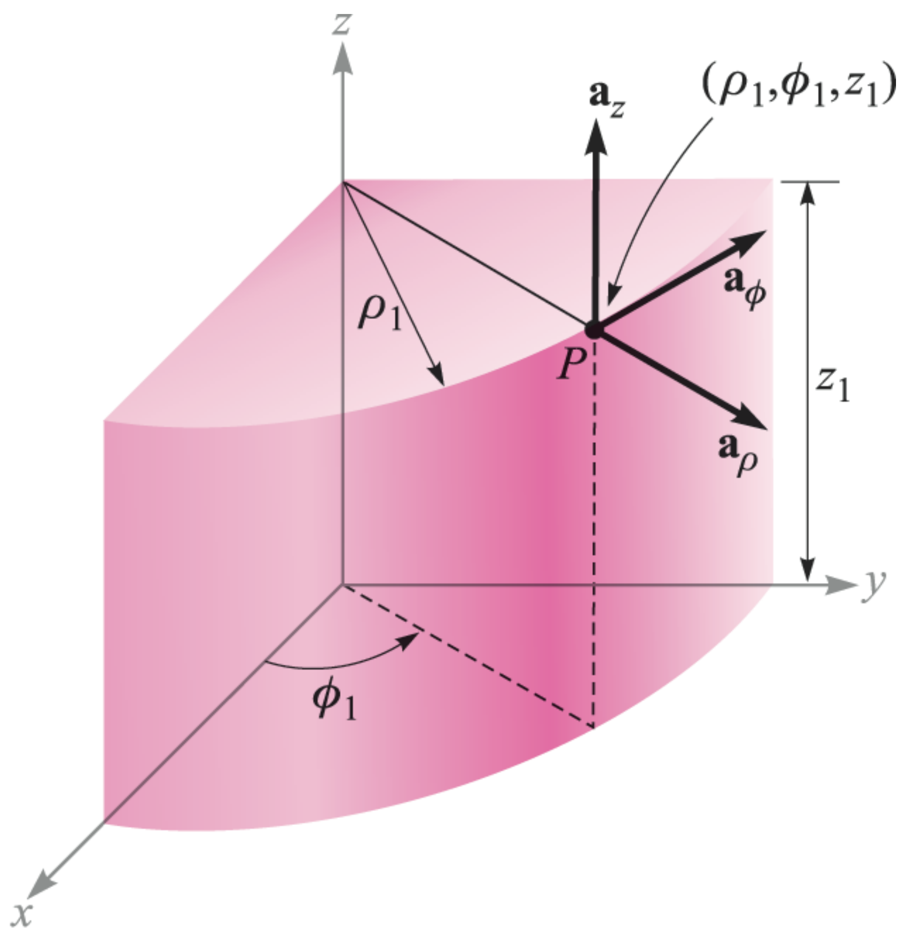
$$\boxed{\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}}$$

Circular Cylindrical Coordinates

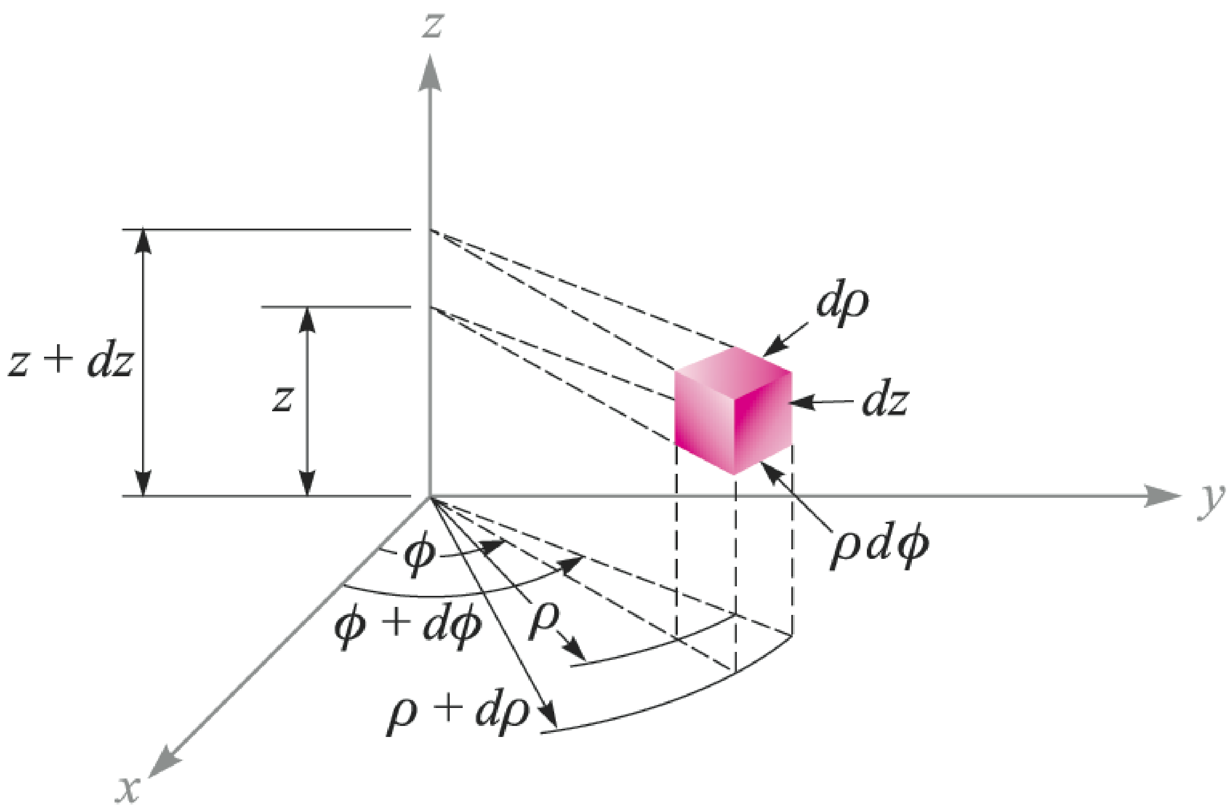
Point P has coordinates
Specified by $P(\rho,\phi,z)$



Orthogonal Unit Vectors in Cylindrical Coordinates

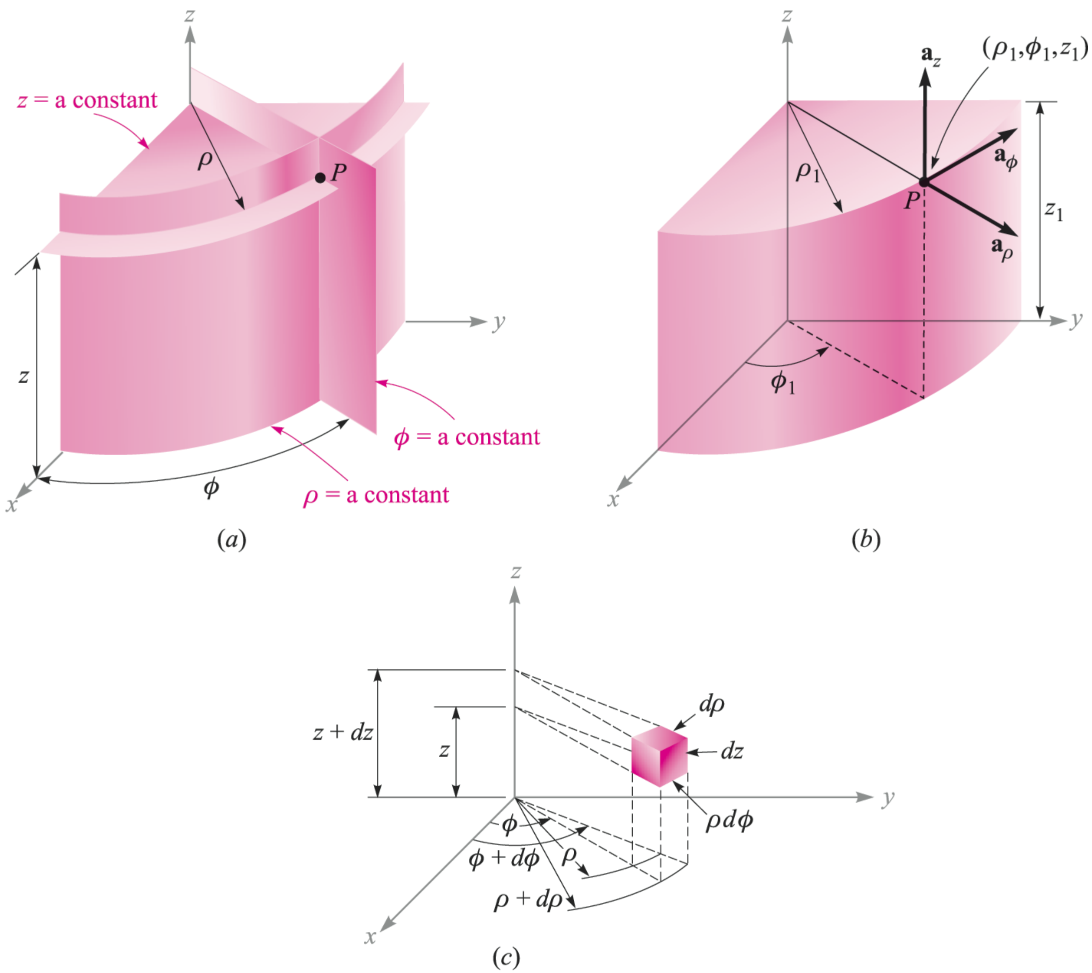


Differential Volume in Cylindrical Coordinates



$$dV = \rho \ \ d\rho \ \ d\phi \ \ dz$$

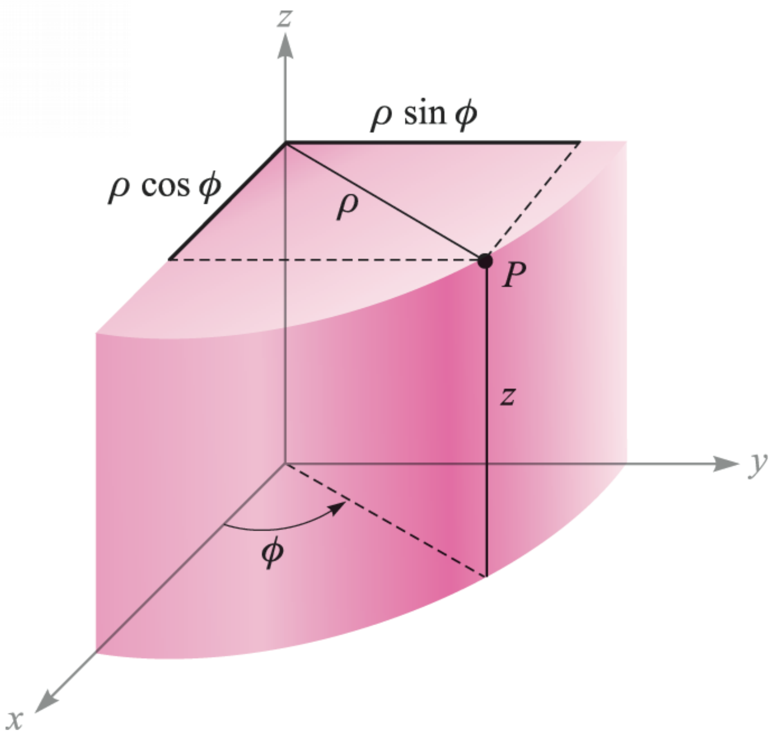
Summary



z = a constant
ρ
P
z
φ
x
y
φ = a constant
ρ = a constant
(a)
a_z
(ρ_1,φ_1,z_1)
ρ_1
a_φ
P
z_1
a_ρ
φ_1
(b)
z + dz
dρ
dz
ρ dφ
φ
φ + dφ
ρ + dρ
(c)

Point Transformations in Cylindrical Coordinates

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$
$$\phi = \tan^{-1} \frac{y}{x}$$
$$z = z$$



$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

Dot Products of Unit Vectors in Cylindrical and Rectangular Coordinate Systems

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x \cdot$	$\cos\phi$	$-\sin$	0
$\mathbf{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z \cdot$	0	0	0

Example

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Start with:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Then:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

$$= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$$

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Finally:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$= y \cos\phi - x \sin\phi = \rho \sin\phi \cos\phi - \rho \cos\phi \sin\phi = 0$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

$$= -y \sin\phi - x \cos\phi = -\rho \sin^2\phi - \rho \cos^2\phi = -\rho$$

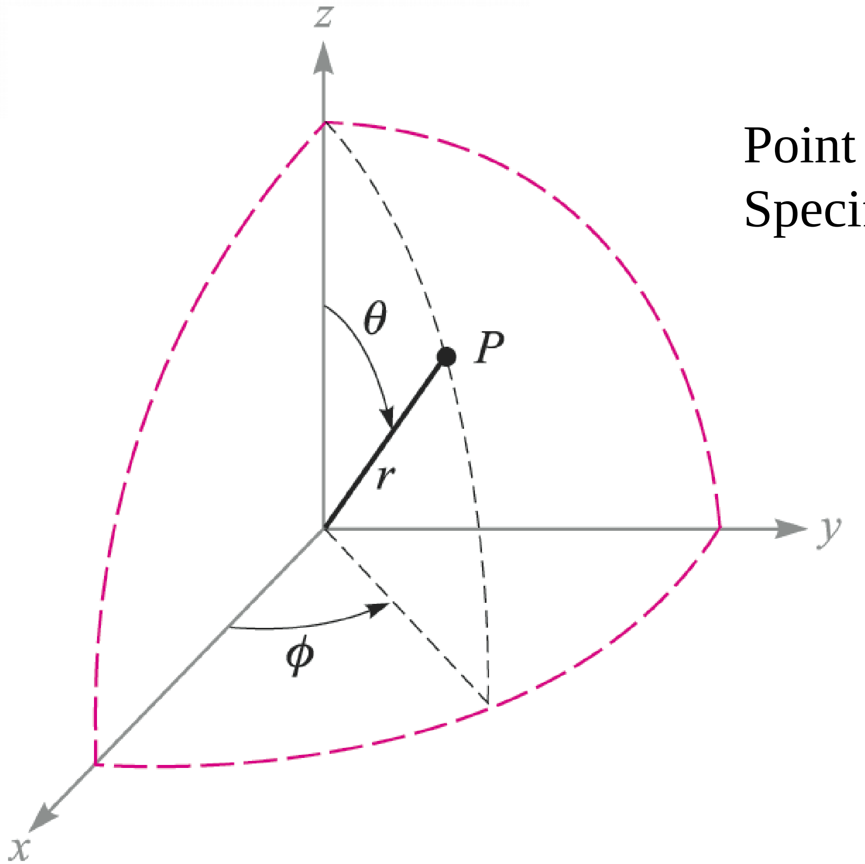
$$\boxed{\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z}$$

Spherical Coordinates

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0)$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0^\circ \leq \theta \leq 180^\circ)$$

$$\phi = \tan^{-1} \frac{y}{x}$$



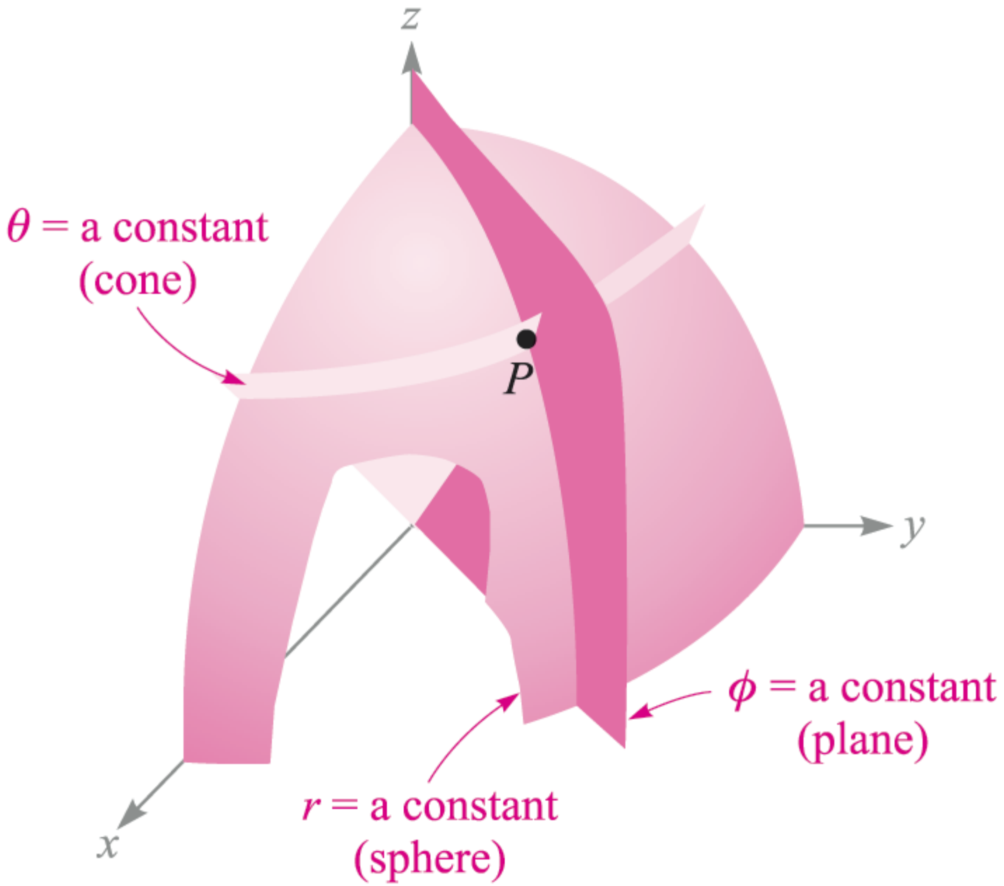
Point P has coordinates
Specified by $P(r,\theta,\phi)$

$$x = r \sin\theta \cos\phi$$

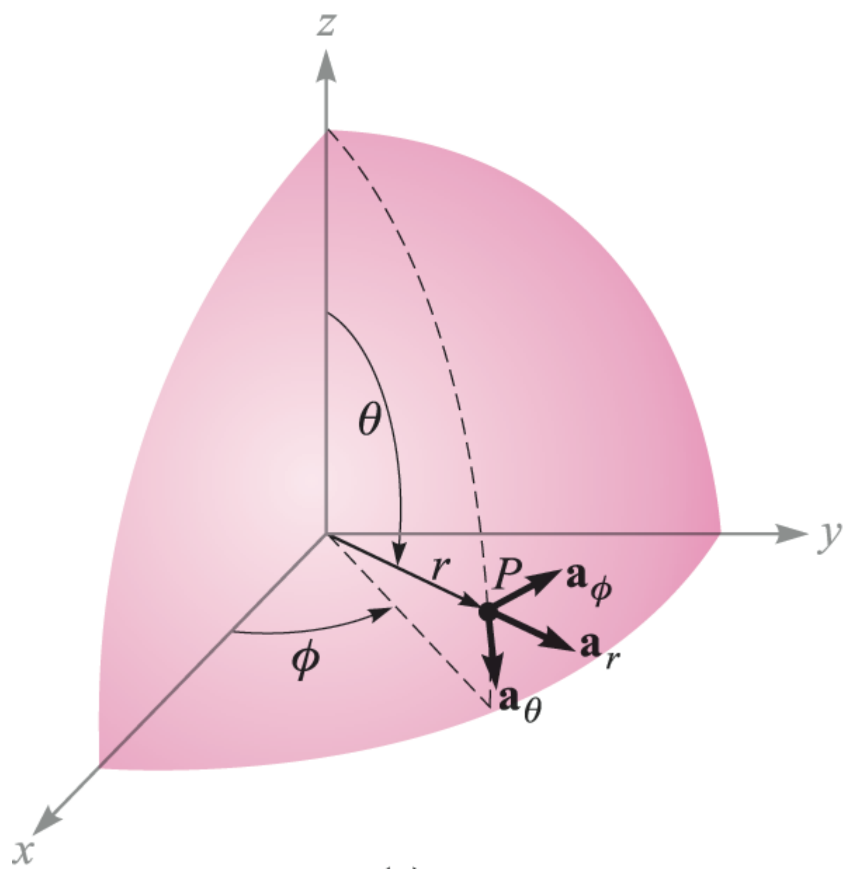
$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Constant Coordinate Surfaces in Spherical Coordinates

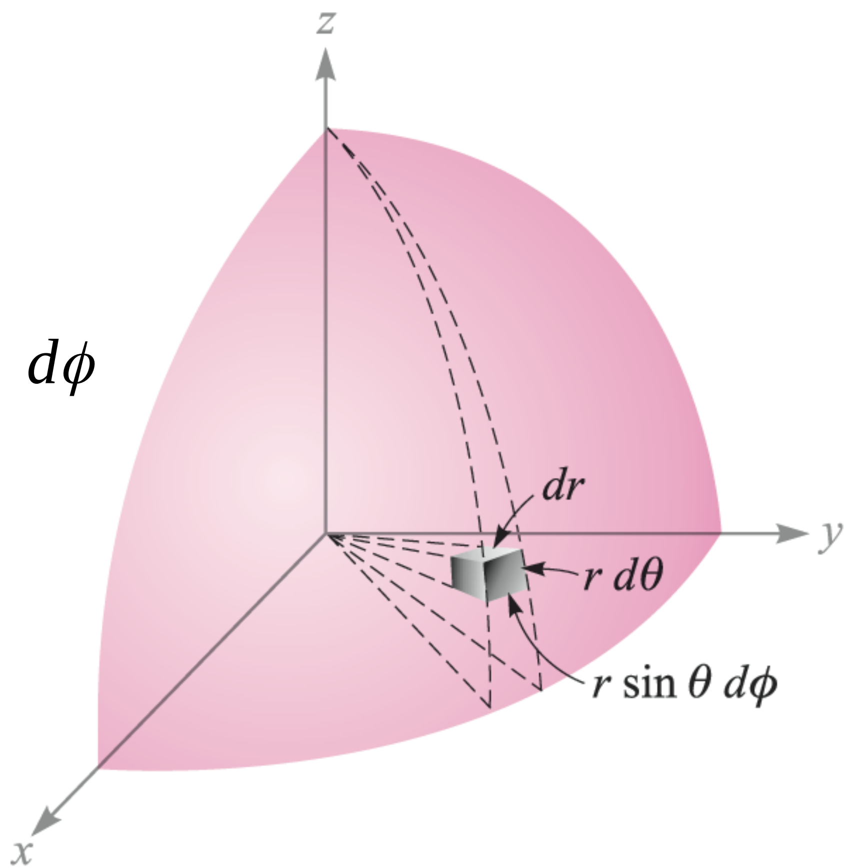


Unit Vector Components in Spherical Coordinates



Differential Volume in Spherical Coordinates

$$dV = r^2 \sin\theta \quad dr d\theta \; d\phi$$



Dot Products of Unit Vectors in the Spherical and Rectangular Coordinate Systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

Example: Vector Component Transformation

Transform the field, $\mathbf{G} = (xz/y)\mathbf{a}_x$, into spherical coordinates and components

$$
\begin{aligned}
G_r &= \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_r = \frac{xz}{y} \sin\theta \cos\phi \\
&= r \sin\theta \cos\theta \frac{\cos^2\phi}{\sin\phi} \\
G_\theta &= \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\theta = \frac{xz}{y} \cos\theta \cos\phi \\
&= r \cos^2\theta \frac{\cos^2\phi}{\sin\phi} \\
G\phi &= \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi) \\
&= -r \cos\theta \cos\phi
\end{aligned}
$$

$$\mathbf{G} = r \cos\theta \cos\phi \,(\sin\theta \cot\phi \,\mathbf{a}_r + \cos\theta \cot\phi \,\mathbf{a}_\theta - \mathbf{a}_\phi)$$

Summary Illustrations